

The Limit of the Empirical Measure of the Product of Two Independent Mallows Permutations

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Abstract

The Mallows measure is a probability measure on S_n where the probability of a permutation π is proportional to $q^{l(\pi)}$ with $q > 0$ being a parameter and $l(\pi)$ the number of inversions in π . We show the convergence of the random empirical measure of the product of two independent permutations drawn from the Mallows measure, when q is a function of n and $n(1 - q)$ has limit in \mathbb{R} as $n \rightarrow \infty$.

1 Introduction

1.1 Background

Definition 1.1. Given $\pi \in S_n$, the inversion set of π is defined by

$$\text{Inv}(\pi) := \{(i, j) : 1 \leq i < j \leq n \text{ and } \pi(i) > \pi(j)\},$$

and the inversion number of π , denoted by $l(\pi)$, is defined to be the cardinality of $\text{Inv}(\pi)$.

The Mallows measure on S_n is introduced by Mallows in [5]. For $q > 0$, the (n, q) - Mallows measure on S_n is given by

$$\mu_{n,q}(\pi) := \frac{q^{l(\pi)}}{Z_{n,q}},$$

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where $Z_{n,q}$ is the normalizing constant. In other words, under the Mallows measure with parameter $q > 0$, the probability of a permutation π is proportional to $q^{l(\pi)}$.

Mallows measure has been used in modeling ranked and partially ranked data (see, e.g., [3], [4], [6]). In [8], Starr proves the convergence of the empirical measure of Mallows permutation in the regime where $\lim_{n \rightarrow \infty} n(1-q)$ exists. In that paper, Starr makes use of the mean field theory and evaluates the density of the limit distribution as the solution to an integrable PDE. In this paper, we establish a similar result for the empirical measure of the product of two independent Mallows permutations. Here the product of two permutation is taken within the symmetric group S_n , and our proof takes an entirely different approach.

1.2 Results

The following theorem is the one dimensional analog of Theorem 1.1 in [8]. It says that, in the regime of Mallows measure where $\lim_{n \rightarrow \infty} n(1-q_n)$ exists, the distribution of $\frac{\pi(a_n)}{n}$ converges in distribution to a probability measure with explicit density, where $\{a_n\}$ is a sequence of indices such that $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists.

Theorem 1.2. *Suppose that $\{q_n\}_{n=1}^{\infty}$ is a sequence such that the limit $\beta = \lim_{n \rightarrow \infty} n(1-q_n)$ exists. Suppose $\{a_n\}$ is a sequence such that $\lim_{n \rightarrow \infty} \frac{a_n}{n} = a$, where $a \in [0, 1]$ and $a_n \in [n]$. Then,*

$$\mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \in (\cdot) \right) \xrightarrow{d} v.$$

Here v is the probability measure on $[0, 1]$ with density $f(y) = u(a, y, \beta)$, where

$$u(x, y, \beta) := \frac{(\beta/2) \sinh(\beta/2)}{(e^{\beta/4} \cosh(\beta[x-y]/2) - e^{-\beta/4} \cosh(\beta[x+y-1]/2))^2} \quad (1)$$

if $\beta \neq 0$, and $u(x, y, 0) := 1$.

Theorem 1.2 is a major step in proving Theorem 1.3, which shows the convergence of the empirical measure defined by the product of two independent Mallows distributed permutations.

Theorem 1.3. *Suppose that $\{q_n\}_{n=1}^\infty$ and $\{q'_n\}_{n=1}^\infty$ are two sequences such that $\lim_{n \rightarrow \infty} n(1-q_n) = \beta$ and $\lim_{n \rightarrow \infty} n(1-q'_n) = \gamma$, with $\beta, \gamma \in \mathbb{R}$. Let \mathbb{P}_n denote the probability measure on $S_n \times S_n$ such that $\mathbb{P}_n((\pi, \tau)) = \mu_{n,q_n}(\pi) \cdot \mu_{n,q'_n}(\tau)$, i. e. \mathbb{P}_n is the product measure of μ_{n,q_n} and μ_{n,q'_n} . Let $\tau \circ \pi$ denote the product of τ and π in S_n with $\tau \circ \pi(i) = \tau(\pi(i))$. Then, for any $\epsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left(\left| \frac{1}{n} \sum_{i=1}^n f \left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n} \right) - \int_0^1 \int_0^1 f(x, y) \rho(x, y) dx dy \right| > \epsilon \right) = 0$$

for every continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, with

$$\rho(x, y) := \int_0^1 u(x, t, \beta) \cdot u(t, y, \gamma) dt,$$

where $u(x, y, \beta)$ is defined in (1).

2 Preliminaries

Let μ be a probability measure on the Borel σ -field \mathcal{B}_Σ . We use the convention that $\mu(f) = \int_\Sigma f d\mu$, for any measurable function f . For any $\pi \in S_n$, let L_π denote the empirical measure induced by π , that is,

$$L_\pi(R) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_R \left(\frac{i}{n}, \frac{\pi(i)}{n} \right),$$

for any $R \in \mathcal{B}_{[0,1] \times [0,1]}$. Here $\mathbf{1}_R(x, y)$ denotes the indicator function of R . Hence, for any measurable function f ,

$$L_\pi(f) = \frac{1}{n} \sum_{i=1}^n f \left(\frac{i}{n}, \frac{\pi(i)}{n} \right).$$

For any $\pi \in S_n$, Let $\mathbf{z}(\pi) := \{(\frac{i}{n}, \frac{\pi(i)}{n})\}_{i \in [n]}$ denote the set of n points in $[0, 1] \times [0, 1]$ defined by π . Conversely, for any n points $V := \{(x_i, y_i)\}_{i \in [n]}$ such that $i \neq j$ implies $x_i \neq x_j$ and $y_i \neq y_j$, we can define a permutation $\pi \in S_n$ as follows. Without loss of generality, assuming $x_1 < \dots < x_n$, define

$$\pi(i) := |\{j \in [n] : y_j \leq y_i\}|.$$

We will use $\Phi(V)$ to denote the permutation induced by V as above. Similarly, we define the number of inversions of a collection points as follows,

$$l(V) := |\{(i, j) : (x_i - x_j)(y_i - y_j) < 0 \text{ and } i < j\}|.$$

Note that the definition of the number of inversions of a collection of points is consistent with the definition of inversion of permutation in the sense that, for any $\pi \in S_n$,

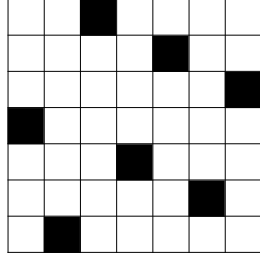
$$l(\pi) = l(\mathbf{z}(\pi)) \quad \text{and} \quad l(V) = l(\Phi(V)).$$

Definition 2.1. For any $\pi \in S_n$ and $i \in [n]$, define

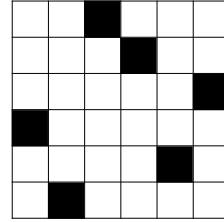
$$\pi^{(i)} := \Phi\left(\left\{\left(\frac{j}{n}, \frac{\pi(j)}{n}\right) : j \neq i\right\}\right) \quad \text{and} \quad Q(\pi, i) := \{\tau \in S_n : \tau^{(i)} = \pi^{(i)}\}.$$

In other words, $\pi^{(i)}$ denotes the permutation in S_{n-1} which is induced from π at those indices other than i , and $Q(\pi, i)$ contains those permutations in S_n each of which has the same relative ordering as π at those indices other than i .

The definition above is best understood when we represent a permutation by a grid of tiles. Specifically, for any $\pi \in S_n$, define an $n \times n$ grid of tiles such that the tile at j -th row and i -th column is black if only if $\pi(i) = j$. Here we index the row number from bottom to top, i.e. the bottom row is indexed as the first row. For example, the grid representations of $\pi = (4, 1, 7, 3, 6, 2, 5)$ and $\pi^{(4)} = (3, 1, 6, 5, 2, 4)$ are shown in the following figures.



$$\pi = (4, 1, 7, 3, 6, 2, 5)$$



$$\pi^{(4)} = (3, 1, 6, 5, 2, 4)$$

Note that the grid representation of $\pi^{(i)}$ can be easily obtained by deleting the i -th column and $\pi(i)$ -th row from the grid of π . Also, the grid representations

of those permutations other than π in $Q(\pi, i)$ can be obtained by removing and reinserting the $\pi(i)$ -th row into the grid of π . For example, it can be easily verified that $\tau = (3, 1, 7, 6, 5, 2, 4) \in Q(\pi, 4)$. The grid representation of τ can be obtained by removing the third row from the grid of π and reinserting it between the sixth row and seventh row of the grid of π (see Figure 1).

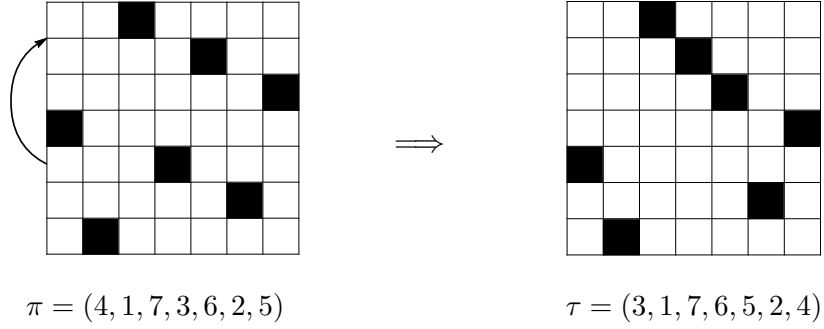


Figure 1

From this definition, it can be seen that $|Q(\pi, i)| = n$ for any $\pi \in S_n$. Also, for any $\pi, \tau \in S_n$, we have either $Q(\pi, i) = Q(\tau, i)$ or $Q(\pi, i) \cap Q(\tau, i) = \emptyset$.

Proposition 2.2. *For any $\pi, \tau \in Q(\pi, i)$, with $\pi(i) = j < k = \tau(i)$, it holds that*

$$\begin{aligned} l(\tau) - l(\pi) &= |\{t > i : j + 1 \leq \pi(t) \leq k\}| - |\{t < i : j + 1 \leq \pi(t) \leq k\}| \\ &= |\{t > i : j \leq \tau(t) \leq k - 1\}| - |\{t < i : j \leq \tau(t) \leq k - 1\}|. \end{aligned}$$

Proof. This result can be easily seen from the grid representations of π and τ . Note that an inversion in a permutation corresponds to a pair of black tiles such that one tile is located to the southeast of the other. Hence, by the discussion above, we only need to count the change of the number of those pairs when we reinsert the j -th row of π 's grid to get the grid form of τ . Specifically, we only need to consider those pairs which contain the black tile on the i -th column.

Taking the same example above, $l(\tau) - l(\pi)$ is equal to the difference of the number of black tiles within the rectangles A and B (see Figure 2). This is because, each of those black tiles in rectangle A forms an inversion with the black tile in the fourth column in the grid representation of π but not in that of τ , whereas the opposite holds for those black tiles in the rectangle B . \square

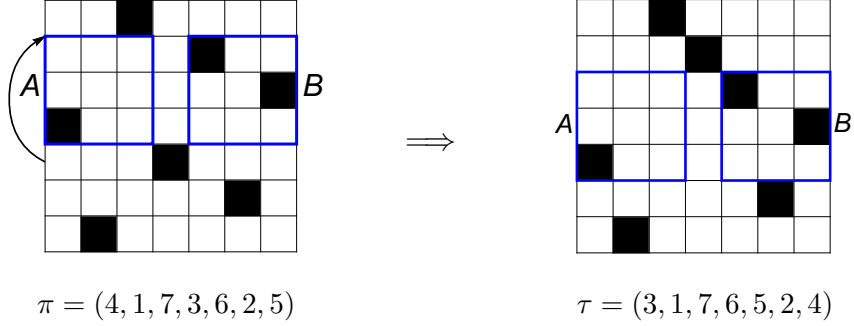


Figure 2

Definition 2.3. For any $\pi \in S_n$, let $\pi^r \in S_n$ denote the reversal of π which is defined by $\pi^r(i) := \pi(n+1-i)$ for any $i \in [n]$. Let π^{-1} denote the inverse of π in the symmetric group S_n .

One property of Mallows permutation is the following proposition (cf. Lemma 2.2 in [2]).

Proposition 2.4. For any $n \geq 1$ and $q > 0$, if $\pi \sim \mu_{n,q}$ then $\pi^r \sim \mu_{n,1/q}$ and $\pi^{-1} \sim \mu_{n,q}$.

3 Proof of Theorem 1.2

In Starr's paper [8], he proves the following result showing that the random empirical measure defined by the points $\{(\frac{i}{n}, \frac{\pi(i)}{n})\}_{i \in [n]}$ converges in probability to a non-random probability measure with an explicit density.

Theorem 3.1. Suppose that $(q_n)_{n=1}^\infty$ is a sequence such that the limit $\beta = \lim_{n \rightarrow \infty} n(1 - q_n)$ exists. For any $\epsilon > 0$ and any continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mu_{n,q_n} \left(\left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}, \frac{\pi(i)}{n}\right) - \int_{[0,1] \times [0,1]} f(x,y) u(x,y) \, dx dy \right| > \epsilon \right) = 0,$$

where

$$u(x,y,\beta) = \frac{(\beta/2) \sinh(\beta/2)}{(e^{\beta/4} \cosh(\beta[x-y]/2) - e^{-\beta/4} \cosh(\beta[x+y-1]/2))^2}.$$

We will sometimes omit the third argument and simply use $u(x, y)$ to denote $u(x, y, \beta)$, if no confusion arises from the context. We use the symbol u_β or u to denote the measure on $[0, 1] \times [0, 1]$ which has density $u(x, y, \beta)$ with respect to the Lebesgue measure λ .

Theorem 1.2 is a one dimensional analog of the result above. It is unknown to us whether Theorem 1.2 can be obtained directly from Starr's result. In the remainder of this section, we prove a sequence of technical lemmas to show Theorem 1.2. The following lemma says that the result of Theorem 3.1 also holds when f is an indicator function of any rectangle.

Lemma 3.2. *Under the same conditions as in Theorem 3.1, for any $\epsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mu_{n, q_n} \left(\left| \frac{1}{n} \sum_{i=1}^n \mathbb{1}_R \left(\frac{i}{n}, \frac{\pi(i)}{n} \right) - \int_R u(x, y) \, dx dy \right| > \epsilon \right) = 0,$$

for any $R = [x_1, x_2] \times [y_1, y_2] \subset [0, 1] \times [0, 1]$.

Proof. First we show that for any $R = [x_1, x_2] \times [y_1, y_2]$ and any $\epsilon > 0$, when n is sufficiently large,

$$L_\pi(R) < \min(x_2 - x_1, y_2 - y_1) + \frac{\epsilon}{24} \quad (2)$$

for any $\pi \in S_n$. Let $s := \min(x_2 - x_1, y_2 - y_1)$. For any $\pi \in S_n$, we have,

$$\left| \left\{ i : \left(\frac{i}{n}, \frac{\pi(i)}{n} \right) \in R \right\} \right| \leq ns + 1$$

since, of the points in $\left\{ \left(\frac{i}{n}, \frac{\pi(i)}{n} \right) \right\}$, there is one and only one point on each line $x = \frac{i}{n}$ or $y = \frac{j}{n}$. Hence, $L_\pi(R) \leq s + \frac{1}{n}$ for any $\pi \in S_n$. We can choose n large enough such that $\frac{1}{n} < \frac{\epsilon}{24}$.

Next, for any $\delta > 0$, let $R_\delta := (x_1 - \delta, x_2 + \delta) \times (y_1 - \delta, y_2 + \delta)$. Let $D := R_\delta - R$. Then, it is easily seen that D can be covered by four rectangles each of whose smaller side is no greater than δ .

For any $\delta > 0$, by Urysohn's lemma (cf. 12.1 in [7]), we can choose a continuous function $f_{R_\delta}(x, y)$, such that,

$$\begin{cases} f_{R_\delta}(x, y) = 1 & \text{if } (x, y) \in R \\ f_{R_\delta}(x, y) = 0 & \text{if } (x, y) \notin R_\delta \\ 0 \leq f_{R_\delta}(x, y) \leq 1 & \text{if } (x, y) \in D. \end{cases}$$

By the triangle inequality, we have

$$\begin{aligned} & |L_\pi(R) - u(R)| > \epsilon \\ \Rightarrow & |L_\pi(f_{R_\delta}) - L_\pi(R)| + |u(f_{R_\delta}) - u(R)| + |L_\pi(f_{R_\delta}) - u(f_{R_\delta})| > \epsilon. \end{aligned} \quad (3)$$

If we choose $\delta < \frac{\epsilon}{24}$, by (2), we have,

$$|L_\pi(f_{R_\delta}) - L_\pi(R)| \leq L_\pi(R_\delta) - L_\pi(R) = L_\pi(D) < 4 \left(\frac{\epsilon}{24} + \frac{\epsilon}{24} \right) = \frac{\epsilon}{3}$$

for any $\pi \in S_n$, when n is sufficiently large. Since u is absolutely continuous with respect to the Lebesgue measure, we may choose δ small enough such that

$$|u(f_{R_\delta}) - u(R)| \leq u(D) < \frac{\epsilon}{3}.$$

Then by (3), for sufficiently large n , we have,

$$|L_\pi(R) - u(R)| > \epsilon \quad \Rightarrow \quad |L_\pi(f_{R_\delta}) - u(f_{R_\delta})| > \frac{\epsilon}{3}.$$

Thus,

$$\mu_{n,q} \left(|L_\pi(R) - u(R)| > \epsilon \right) \leq \mu_{n,q} \left(|L_\pi(f_{R_\delta}) - u(f_{R_\delta})| > \frac{\epsilon}{3} \right).$$

The lemma follows by Theorem 3.1. \square

The following property of Mallows distributed permutations will be used in later proofs. It says that in a Mallows permutation, the relative chance that $\pi(i)$ takes two different values can be bounded in terms of the difference of those two values.

Lemma 3.3. *For any $1 \leq i, s, t \leq n$ and $q > 0$,*

$$\min(q^d, q^{-d}) \leq \frac{\mu_{n,q}(\pi(i) = s)}{\mu_{n,q}(\pi(i) = t)} \leq \max(q^d, q^{-d}),$$

where $d = |s - t|$.

Proof. Suppose $0 < q < 1$. We claim it suffices to show that

$$q \leq \frac{\mu_{n,q}(\pi(i) = j + 1)}{\mu_{n,q}(\pi(i) = j)} \leq \frac{1}{q}, \quad (4)$$

for any $j \in [n-1]$. This follows since by taking the reciprocal of (4), we get

$$q \leq \frac{\mu_{n,q}(\pi(i) = j)}{\mu_{n,q}(\pi(i) = j+1)} \leq \frac{1}{q},$$

and the lemma follows by induction on d .

Consider the bijection T_j on S_n : $\pi \rightarrow (j, j+1) \circ \pi$. Here \circ denotes the group operator of S_n , and $(j, j+1)$ denotes the transposition of j and $j+1$. Specifically, for any $i \in [n]$

$$T_j(\pi)(i) = \begin{cases} j & \text{if } \pi(i) = j+1 \\ j+1 & \text{if } \pi(i) = j \\ \pi(i) & \text{otherwise.} \end{cases}$$

From the definition, it is not hard to see that $|l(\pi) - l(T_j(\pi))| = 1$, for any $\pi \in S_n$. Hence,

$$q \leq \frac{\mu_{n,q}(T_j(\pi))}{\mu_{n,q}(\pi)} \leq \frac{1}{q}. \quad (5)$$

Let $A_{i,j} := \{\pi \in S_n : \pi(i) = j\}$. For any fixed $i \in [n]$, T_j is also a bijection of $A_{i,j}$ and $A_{i,j+1}$. Hence,

$$\frac{\mu_{n,q}(\pi(i) = j+1)}{\mu_{n,q}(\pi(i) = j)} = \frac{\sum_{\pi \in A_{i,j}} \mu_{n,q}(T_j(\pi))}{\sum_{\pi \in A_{i,j}} \mu_{n,q}(\pi)}, \quad (6)$$

and (4) follows from (5) and (6).

For the case $q > 1$, the proof is similar. The lemma clearly also holds when $q = 1$, which corresponds to the uniform measure on S_n . \square

The following result establishes some bounds on the probability of a point in a Mallows permutation appearing in an interval.

Lemma 3.4. *Suppose that $(q_n)_{n=1}^\infty$ is a sequence such that the limit $\beta = \lim_{n \rightarrow \infty} n(1 - q_n)$ exists. For any sequence $\{a_n\}$ with $a_n \in [n]$ and any $0 \leq y_1 < y_2 \leq 1$,*

$$\limsup_{n \rightarrow \infty} \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \in [y_1, y_2] \right) \leq (y_2 - y_1) e^{|\beta|}, \quad (7)$$

$$\liminf_{n \rightarrow \infty} \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \in (y_1, y_2) \right) \geq (y_2 - y_1) e^{-|\beta|}. \quad (8)$$

Proof. Here we only prove the case $\beta \geq 0$. The case $\beta < 0$ follows from the same argument. We also assume that $y_2 - y_1 < 1$, since the case $y_0 = 0, y_1 = 1$ can be verified easily. Since $\lim_{n \rightarrow \infty} n(1 - q_n) = \beta$ and $\lim_{n \rightarrow \infty} \frac{n \log q_n}{n(1 - q_n)} = -1$, we have

$$\lim_{n \rightarrow \infty} q_n^n = \lim_{n \rightarrow \infty} e^{n \log q_n} = e^{-\beta}.$$

Thus, for any $\delta > 1$, there exists $N > 0$ such that $q_n^n \in \left(\frac{e^{-\beta}}{\delta}, \delta e^{-\beta}\right)$, when $n > N$. By Lemma 3.3, for any $n > N$ and any $i, s, t \in [n]$

$$\frac{\mu_{n,q_n}(\pi(i) = s)}{\mu_{n,q_n}(\pi(i) = t)} \leq \max\left(q_n^n, \frac{1}{q_n^n}\right) < \delta e^\beta. \quad (9)$$

Let $d = y_2 - y_1$ and $p_n = \min_{\{t: \frac{t}{n} \notin [y_1, y_2]\}} (\mu_{n,q_n}(\pi(a_n) = t))$. Note that the set $\{t: \frac{t}{n} \notin [y_1, y_2]\}$ is nonempty for sufficiently large n . Then, by (9) and the fact that,

$$\left|\left\{k \in [n] : \frac{k}{n} \in [y_1, y_2]\right\}\right| \leq nd + 1, \quad \left|\left\{k \in [n] : \frac{k}{n} \notin [y_1, y_2]\right\}\right| \geq n(1 - d) - 1,$$

we have,

$$\begin{aligned} \mu_{n,q_n}\left(\frac{\pi(a_n)}{n} \in [y_1, y_2]\right) &< (nd + 1)\delta e^\beta p_n, \\ \mu_{n,q_n}\left(\frac{\pi(a_n)}{n} \notin [y_1, y_2]\right) &\geq (n(1 - d) - 1)p_n. \end{aligned}$$

Hence,

$$\begin{aligned} \mu_{n,q_n}\left(\frac{\pi(a_n)}{n} \in [y_1, y_2]\right) &< \frac{(nd + 1)\delta e^\beta}{(n(1 - d) - 1) + (nd + 1)\delta e^\beta} \\ &< \frac{(nd + 1)\delta e^\beta}{(n(1 - d) - 1) + (nd + 1)} \\ &= \frac{(nd + 1)\delta e^\beta}{n}, \end{aligned}$$

and (7) follows since δ can be chosen arbitrarily close to 1. Similarly, to prove (8), define $p'_n = \min_{\{t: \frac{t}{n} \in (y_1, y_2)\}} (\mu_{n,q_n}(\pi(a_n) = t))$. Then, by (9) and the fact that,

$$\left|\left\{k \in [n] : \frac{k}{n} \in (y_1, y_2)\right\}\right| \geq nd - 1, \quad \left|\left\{k \in [n] : \frac{k}{n} \notin (y_1, y_2)\right\}\right| \leq n(1 - d) + 1,$$

we have

$$\begin{aligned}\mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \in (y_1, y_2) \right) &\geq (nd - 1)p'_n, \\ \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \notin (y_1, y_2) \right) &< (n(1 - d) + 1)\delta e^\beta p'_n.\end{aligned}$$

Hence,

$$\begin{aligned}\mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \in (y_1, y_2) \right) &> \frac{(nd - 1)}{(n(1 - d) + 1)\delta e^\beta + (nd - 1)} \\ &> \frac{(nd - 1)}{(n(1 - d) + 1)\delta e^\beta + (nd - 1)\delta e^\beta} \\ &= \frac{(nd - 1)e^{-\beta}}{n\delta}.\end{aligned}$$

And (8) follows since δ can be chosen arbitrarily close to 1. □

In the next two lemmas, we introduce some properties of the density function $u(x, y, \beta)$ defined in Theorem 3.1.

Lemma 3.5. *With $u(x, y, \beta)$ defined as in Theorem 1.2, we have*

$$\begin{aligned}\int_0^1 u(x, y, \beta) dx &= 1, \quad \forall y \in [0, 1], \\ \int_0^1 u(x, y, \beta) dy &= 1, \quad \forall x \in [0, 1].\end{aligned}$$

Proof. Since $\cosh(x)$ is an even function, $u(x, y, \beta)$ is symmetric with respect to the line $y = x$. That is

$$u(x, y, \beta) = u(y, x, \beta), \quad \forall x, y \in [0, 1].$$

Hence we only need to show the first identity. By [8, Corollary 6.2],

$$\frac{\partial^2 \ln u(x, y, \beta)}{\partial x \partial y} = 2\beta u(x, y, \beta). \tag{10}$$

Therefore, we have

$$\int_0^1 u(x, y, \beta) dx = \frac{1}{2\beta} \left(\frac{\partial \ln u(1, y, \beta)}{\partial y} - \frac{\partial \ln u(0, y, \beta)}{\partial y} \right). \quad (11)$$

Next, by direct calculation, we have

$$u(1, y, \beta) = \frac{(\beta/2) \sinh(\beta/2)}{\left(\frac{1}{2}e^{-\frac{\beta}{4}}(e^\beta - 1)e^{-\frac{\beta y}{2}}\right)^2} = \frac{\beta e^{\beta y}}{e^\beta - 1}, \quad (12)$$

$$u(0, y, \beta) = \frac{(\beta/2) \sinh(\beta/2)}{\left(\frac{1}{2}e^{\frac{\beta}{4}}(1 - e^{-\beta})e^{\frac{\beta y}{2}}\right)^2} = \frac{\beta e^{-\beta y}}{1 - e^{-\beta}}. \quad (13)$$

Hence, we get

$$\frac{\partial \ln u(1, y, \beta)}{\partial y} = \beta \quad \text{and} \quad \frac{\partial \ln u(0, y, \beta)}{\partial y} = -\beta.$$

By (11), the lemma follows. \square

In the remainder of this section, we will simply use $u(x, y)$ to denote $u(x, y, \beta)$.

Lemma 3.6. *For any $0 \leq a, c, d \leq 1$,*

$$-\beta \int_c^d \left(-\int_0^a u(x, y) dx + \int_a^1 u(x, y) dx \right) dy = \ln \frac{u(a, d)}{u(a, c)}.$$

Proof. For fixed $c, d \in [0, 1]$, define $f(a)$ to be the left-hand side of the identity and $g(a)$ to be the right-hand side of the identity. Then, by Lemma 3.5 and (13), we have

$$f(0) = g(0) = \beta(c - d).$$

Hence, to prove the identity it suffices to show $f'(a) = g'(a)$ for any $a \in (0, 1)$. Since $u(x, y)$ is bounded on $[0, 1] \times [0, 1]$, we can change the order of integral and differentiation in the following,

$$\begin{aligned} f'(a) &= -\beta \frac{\partial}{\partial a} \int_c^d \left(-\int_0^a u(x, y) dx + \int_a^1 u(x, y) dx \right) dy \\ &= -\beta \int_c^d \frac{\partial}{\partial a} \left(-\int_0^a u(x, y) dx + \int_a^1 u(x, y) dx \right) dy \end{aligned}$$

$$\begin{aligned}
&= -\beta \int_c^d (-u(a, y) - u(a, y)) \, dy \\
&= 2\beta \int_c^d u(a, y) \, dy.
\end{aligned}$$

By (10), $\frac{\partial \ln u(x, y)}{\partial x}$ is the anti-derivative of $2\beta u(x, y)$ with respect to y . Thus we have

$$g'(a) = \frac{\partial \ln u(a, d)}{\partial a} - \frac{\partial \ln u(a, c)}{\partial a} = 2\beta \int_c^d u(a, y) \, dy.$$

□

Lemma 3.7. *In the context of Lemma 3.4, suppose $\{a_n\}_{n \geq 1}$ is a sequence such that $\lim_{n \rightarrow \infty} \frac{a_n}{n} = a$, where $a_n \in [n]$. For any $0 \leq y_1 < y_2 < 1$,*

$$\lim_{\epsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \left| \frac{\mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (y_2, y_2 + \epsilon) \right)}{\mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (y_1, y_1 + \epsilon) \right)} - \frac{u(a, y_2)}{u(a, y_1)} \right| = 0. \quad (14)$$

For any $0 < y_1 < y_2 \leq 1$,

$$\lim_{\epsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \left| \frac{\mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (y_2 - \epsilon, y_2) \right)}{\mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (y_1 - \epsilon, y_1) \right)} - \frac{u(a, y_2)}{u(a, y_1)} \right| = 0. \quad (15)$$

Proof. Here we only prove (14), since (15) follows from the similar argument. To prove (14), we need to show that for any $\eta > 0$, there exists $\epsilon_0 > 0$ such that for any fixed $\epsilon < \epsilon_0$, there exists $N > 0$, which may depend on ϵ , such that for any $n > N$, we have

$$\left| \frac{\mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (y_2, y_2 + \epsilon) \right)}{\mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (y_1, y_1 + \epsilon) \right)} - \frac{u(a, y_2)}{u(a, y_1)} \right| < \eta. \quad (16)$$

First, we define the following two rectangles:

$$R_0 := [0, a] \times [y_1, y_2], \quad R_1 := [a, 1] \times [y_1, y_2].$$

Next define

$$G(n, \lambda) := \{ \pi \in S_n : |L_\pi(A) - u(A)| < \lambda, \text{ for any } A \in \{R_0, R_1\} \}.$$

Let $\overline{G}(n, \lambda) := S_n \setminus G(n, \lambda)$ denote the complement of $G(n, \lambda)$. Then,

$$\overline{G}(n, \lambda) = \cup_{A \in \{R_0, R_1\}} \{\pi \in S_n : |L_\pi(A) - u(A)| \geq \lambda\}.$$

Thus by Lemma 3.2, for any $\epsilon_0 > 0$ and any $\lambda > 0$, we have

$$\lim_{n \rightarrow \infty} \mu_{n, q_n}(\overline{G}(n, \lambda)) = 0. \quad (17)$$

Define

$$GD(n, \lambda) := \{\pi \in S_n : Q(\pi, a_n) \cap G(n, \lambda/2) \neq \emptyset\}.$$

Note that, for any rectangle R and any $\tau, \xi \in Q(\pi, a_n)$,

$$|L_\tau(R) - L_\xi(R)| \leq \frac{1}{n}.$$

Thus, when $n > \frac{2}{\lambda}$, it follows from triangle inequality that

$$GD(n, \lambda) \subset G(n, \lambda). \quad (18)$$

On the other hand, by the definition of $GD(n, \lambda)$ and the fact that, for any $i \in [n]$, $\pi \in Q(\pi, i)$, it follows that

$$G(n, \lambda/2) \subset GD(n, \lambda). \quad (19)$$

Hence by (17) and (19), for any $\lambda > 0$, we have

$$\lim_{n \rightarrow \infty} \mu_{n, q_n}(GD(n, \lambda)) = 1. \quad (20)$$

Next, given $\epsilon \in (0, \epsilon_0)$ where the value of ϵ_0 is to be determined, define

$$A_n := \{\pi \in S_n : \frac{\pi(a_n)}{n} \in (y_1, y_1 + \epsilon)\}, \quad B_n := \{\pi \in S_n : \frac{\pi(a_n)}{n} \in (y_2, y_2 + \epsilon)\}.$$

Then, by Lemma 3.4, when n is sufficiently large, we have

$$\mu_{n, q_n}(A_n) > \frac{\epsilon}{2} e^{-|\beta|}, \quad \mu_{n, q_n}(B_n) > \frac{\epsilon}{2} e^{-|\beta|}.$$

Thus, by (20), there exists an $N_1 > 0$ such that, for any $n > N_1$, we have

$$\left| \frac{\mu_{n, q_n}(B_n \cap GD(n, \lambda))}{\mu_{n, q_n}(A_n \cap GD(n, \lambda))} - \frac{\mu_{n, q_n}(B_n)}{\mu_{n, q_n}(A_n)} \right| < \frac{\eta}{2}.$$

Therefore, to prove (16), it suffices to show that for sufficiently large n , we have

$$\left| \frac{\mu_{n,q_n}(B_n \cap GD(n, \lambda))}{\mu_{n,q_n}(A_n \cap GD(n, \lambda))} - \frac{u(a, y_2)}{u(a, y_1)} \right| < \frac{\eta}{2}. \quad (21)$$

In order to prove (21), we are going to exploit two things. The first one is the fact that $\{Q(\pi, a_n) : \pi \in GD(n, \lambda)\}$ is a partition of $GD(n, \lambda)$. The second is the following,

$$\begin{aligned} \frac{c_i}{d_i} > r, c_i > 0, d_i > 0 \text{ for } \forall i \in [m] &\Rightarrow \frac{\sum_{i=1}^m c_i}{\sum_{i=1}^m d_i} > r, \\ \frac{c_i}{d_i} < r, c_i > 0, d_i > 0 \text{ for } \forall i \in [m] &\Rightarrow \frac{\sum_{i=1}^m c_i}{\sum_{i=1}^m d_i} < r. \end{aligned}$$

Hence, to prove (21), it suffices to show that, for sufficiently large n , we have

$$\left| \frac{\mu_{n,q_n}(B_n \cap Q(\pi, a_n))}{\mu_{n,q_n}(A_n \cap Q(\pi, a_n))} - \frac{u(a, y_2)}{u(a, y_1)} \right| < \frac{\eta}{2}, \quad (22)$$

for any $Q(\pi, a_n) \subset GD(n, \lambda)$. Note that $A_n \cap Q(\pi, a_n)$ is nonempty for any $\pi \in S_n$, when $n > 1/\epsilon$. The strategy to prove (22) is the following, we show that when n is sufficiently large, for any $Q(\pi, a_n) \subset GD(n, \lambda)$ and any $\tau \in B_n \cap Q(\pi, a_n)$, $\xi \in A_n \cap Q(\pi, a_n)$, we have

$$\left| \frac{1}{n} (l(\tau) - l(\xi)) - I \right| < 2\lambda + 4\epsilon + \frac{4}{n}. \quad (23)$$

Here

$$I := \int_{y_1}^{y_2} \left(- \int_0^a u(x, y) dx + \int_a^1 u(x, y) dx \right) dy = u(R_1) - u(R_0).$$

Note that $\frac{\mu_{n,q_n}(\tau)}{\mu_{n,q_n}(\xi)} = q_n^{l(\tau)-l(\xi)}$. Thus, by (23), for any $\tau \in B_n \cap Q(\pi, a_n)$, $\xi \in A_n \cap Q(\pi, a_n)$, we have

$$q_n^{n(I+2\lambda+4\epsilon+4/n)} \leq \frac{\mu_{n,q_n}(\tau)}{\mu_{n,q_n}(\xi)} \leq q_n^{n(I-2\lambda-4\epsilon-4/n)}.$$

Here we assume $0 < q_n < 1$. (The cases $q_n > 1$ and $q_n = 1$ follow by similar argument.) By the definition of A_n, B_n , we have

$$n\epsilon - 1 \leq |A_n \cap Q(\pi, a_n)|, |B_n \cap Q(\pi, a_n)| \leq n\epsilon + 1.$$

Hence we have

$$\frac{n\epsilon - 1}{n\epsilon + 1} q_n^{n(I+2\lambda+4\epsilon+4/n)} \leq \frac{\mu_{n,q_n}(B_n \cap Q(\pi, a_n))}{\mu_{n,q_n}(A_n \cap Q(\pi, a_n))} \leq \frac{n\epsilon + 1}{n\epsilon - 1} q_n^{n(I-2\lambda-4\epsilon-4/n)}.$$

By Lemma 3.6 and the fact that $\lim_{n \rightarrow \infty} q_n^n = e^{-\beta}$ and $\lim_{n \rightarrow \infty} q_n = 1$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n\epsilon - 1}{n\epsilon + 1} q_n^{n(I+2\lambda+4\epsilon+4/n)} &= \frac{u(a, y_2)}{u(a, y_1)} e^{-\beta(2\lambda+4\epsilon)}, \\ \lim_{n \rightarrow \infty} \frac{n\epsilon + 1}{n\epsilon - 1} q_n^{n(I-2\lambda-4\epsilon-4/n)} &= \frac{u(a, y_2)}{u(a, y_1)} e^{\beta(2\lambda+4\epsilon)}. \end{aligned}$$

Thus, we can choose ϵ_0 and λ small enough such that, for any $\epsilon \in (0, \epsilon_0)$, (22) holds for sufficiently large n .

The remaining part of the proof is to show (23). Suppose n is sufficiently large such that $\frac{a_n}{n} \in (a - \epsilon, a + \epsilon)$. Without loss of generality, suppose $\frac{a_n}{n} \in [a, a + \epsilon)$. (The other case can be shown in a similar argument.) By Proposition 2.2, for any $Q(\pi, a_n) \subset GD(n, \lambda)$, and for any $\tau \in B_n \cap Q(\pi, a_n)$, $\xi \in A_n \cap Q(\pi, a_n)$, we have

$$\begin{aligned} & l(\tau) - l(\xi) \\ &= |\{t > a_n : \xi(a_n) < \xi(t) \leq \tau(a_n)\}| - |\{t < a_n : \xi(a_n) < \xi(t) \leq \tau(a_n)\}| \\ &= |\{\frac{t}{n} > \frac{a_n}{n} : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| - |\{\frac{t}{n} < \frac{a_n}{n} : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| \\ &\leq |\{\frac{t}{n} > a : y_1 < \frac{\xi(t)}{n} < y_2 + \epsilon\}| - |\{\frac{t}{n} < a : y_1 + \epsilon \leq \frac{\xi(t)}{n} \leq y_2\}| \\ &= |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (a, 1] \times (y_1, y_2 + \epsilon)\}| \\ &\quad - |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (0, a) \times [y_1 + \epsilon, y_2]\}| \\ &\leq |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (a, 1] \times (y_1, y_2]\}| + (n\epsilon + 1) \\ &\quad - |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (0, a) \times [y_1, y_2]\}| + (n\epsilon + 1) \\ &\leq |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in [a, 1] \times [y_1, y_2]\}| \\ &\quad - |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in [0, a] \times [y_1, y_2]\}| + 2n\epsilon + 4 \\ &= nL_\xi([a, 1] \times [y_1, y_2]) - nL_\xi([0, a] \times [y_1, y_2]) + 2n\epsilon + 4 \\ &= nL_\xi(R_1) - nL_\xi(R_0) + 2n\epsilon + 4. \end{aligned}$$

The first inequality above follows because $\frac{a_n}{n} \geq a$, $\frac{\xi(a_n)}{n} \in (y_1, y_1 + \epsilon)$ and

$\frac{\tau(a_n)}{n} \in (y_2, y_2 + \epsilon)$. The second inequality follows because

$$|\{t \in [n] : \frac{\xi(t)}{n} \in (y_2, y_2 + \epsilon)\}| \leq n\epsilon + 1,$$

$$|\{t \in [n] : \frac{\xi(t)}{n} \in [y_1, y_1 + \epsilon)\}| \leq n\epsilon + 1.$$

The third inequality follows because, since we change $(0, a)$ to $[0, a]$ in the second term, we add two in the end to compensate the possible extra subtraction. Hence, we have

$$\begin{aligned} \frac{1}{n}(l(\tau) - l(\xi)) &\leq L_\xi(R_1) - L_\xi(R_0) + 2\epsilon + \frac{4}{n} \\ &\leq u(R_1) - u(R_0) + 2\lambda + 2\epsilon + \frac{4}{n} \\ &= I + 2\lambda + 2\epsilon + \frac{4}{n}. \end{aligned} \tag{24}$$

Here we use the fact that, by (18), $\xi \in GD(n, \lambda) \subset G(n, \lambda)$. Similarly, to show the lower bound of $l(\tau) - l(\xi)$, we have

$$\begin{aligned} &l(\tau) - l(\xi) \\ &= |\{t > a_n : \xi(a_n) < \xi(t) \leq \tau(a_n)\}| - |\{t < a_n : \xi(a_n) < \xi(t) \leq \tau(a_n)\}| \\ &= |\{\frac{t}{n} > \frac{a_n}{n} : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| - |\{\frac{t}{n} < \frac{a_n}{n} : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| \\ &= |\{\frac{t}{n} > a : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| - |\{\frac{a_n}{n} \geq \frac{t}{n} > a : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| \\ &\quad - |\{\frac{t}{n} < a : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| - |\{a \leq \frac{t}{n} < \frac{a_n}{n} : \frac{\xi(a_n)}{n} < \frac{\xi(t)}{n} \leq \frac{\tau(a_n)}{n}\}| \\ &\geq |\{\frac{t}{n} > a : y_1 + \epsilon \leq \frac{\xi(t)}{n} \leq y_2\}| - (n\epsilon + 1) \\ &\quad - |\{\frac{t}{n} < a : y_1 < \frac{\xi(t)}{n} < y_2 + \epsilon\}| - (n\epsilon + 1) \\ &= |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (a, 1] \times [y_1 + \epsilon, y_2]\}| \\ &\quad - |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (0, a) \times (y_1, y_2 + \epsilon)\}| - 2(n\epsilon + 1) \\ &\geq |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (a, 1] \times [y_1, y_2]\}| - (n\epsilon + 1) \\ &\quad - |\{t : (\frac{t}{n}, \frac{\xi(t)}{n}) \in (0, a) \times (y_1, y_2]\}| - (n\epsilon + 1) - 2(n\epsilon + 1) \\ &= nL_\xi([a, 1] \times [y_1, y_2]) - nL_\xi([0, a] \times [y_1, y_2]) - 4n\epsilon - 4 \\ &= nL_\xi(R_1) - nL_\xi(R_0) - 4n\epsilon - 4. \end{aligned}$$

The first inequality above follows since, by the definition of A_n, B_n , we have $\frac{\xi(a_n)}{n} \in (y_1, y_1 + \epsilon)$, $\frac{\tau(a_n)}{n} \in (y_2, y_2 + \epsilon)$ and, since $\frac{a_n}{n} \in [a, a + \epsilon)$,

$$|\{t \in [n] : \frac{a_n}{n} \geq \frac{t}{n} > a\}| \leq n\epsilon + 1, \quad |\{t \in [n] : a \leq \frac{t}{n} < \frac{a_n}{n}\}| \leq n\epsilon + 1.$$

The second inequality follows because

$$\begin{aligned} |\{t \in [n] : \frac{\xi(t)}{n} \in [y_1, y_1 + \epsilon)\}| &\leq n\epsilon + 1, \\ |\{t \in [n] : \frac{\xi(t)}{n} \in (y_2, y_2 + \epsilon)\}| &\leq n\epsilon + 1. \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{1}{n}(l(\tau) - l(\xi)) &\geq L_\xi(R_1) - L_\xi(R_0) - 4\epsilon - \frac{4}{n} \\ &\geq u(R_1) - u(R_0) - 2\lambda - 4\epsilon - \frac{4}{n} \\ &= I - 2\lambda - 4\epsilon - \frac{4}{n}. \end{aligned} \tag{25}$$

Here again we use the fact that, by (18), $\xi \in GD(n, \lambda) \subset G(n, \lambda)$. The fact that (23) follows from (24) and (25) completes the proof. \square

To complete the proof of Theorem 1.2 we use the following result (cf. 7.2.5 in [1]) and the next two lemmas.

Theorem 3.8. *Let $\{u_n\}_{n \geq 1}$ be a sequence of finite measure on \mathbb{R} . If $\{u_n\}_{n \geq 1}$ is tight, and every weakly convergent subsequence of $\{u_n\}_{n \geq 1}$ converges to the measure v , then $u_n \xrightarrow{d} v$.*

Lemma 3.9. *Suppose $\{a_n\}_{n \geq 1}$ is a sequence such that $\lim_{n \rightarrow \infty} \frac{a_n}{n} = a$, where $a_n \in [n]$, and $\{a_{t_n}\}$ is a subsequence of $\{a_n\}$ such that*

$$\mu_{t_n, q_{t_n}} \left(\frac{\pi(a_{t_n})}{t_n} \in (\cdot) \right) \xrightarrow{d} v.$$

Then the distribution function $F_v(y)$ of the limit probability measure v is absolutely continuous. Here $\mu_{t_n, q_{t_n}} \left(\frac{\pi(a_{t_n})}{t_n} \in (\cdot) \right)$ denotes the probability measure induced by $\frac{\pi(a_{t_n})}{t_n}$ under $\mu_{t_n, q_{t_n}}$.

Proof. For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{4e^{|\beta|}}$. By the definition of absolute continuity, we will show that, for any $\{(y_1, y_2), (y_3, y_4), \dots, (y_{2m-1}, y_{2m})\}$ with $y_{2k-1} < y_{2k}$ and $\sum_{k=1}^m |y_{2k} - y_{2k-1}| < \delta$, we have $\sum_{k=1}^m |F_v(y_{2k}) - F_v(y_{2k-1})| < \epsilon$. Without loss of generality, we may assume that every y_i is a continuous point of $F_v(y)$ with $0 \leq y_i \leq 1$. Since there are at most countably many discontinuity of $F_v(y)$, we can always choose a new set of interval $\{(y'_{2k-1}, y'_{2k})\}$ such that

$F_v(y)$ is continuous at every y'_i , $[y_{2k-1}, y_{2k}] \subset [y'_{2k-1}, y'_{2k}]$ and $\sum_{k=1}^m |y'_{2k} - y'_{2k-1}| < \delta$ still holds. Next, for the simplicity of notation, define

$$v_n := \mu_{t_n, q_{t_n}} \left(\frac{\pi(a_{t_n})}{t_n} \in (\cdot) \right). \quad (26)$$

By Lemma 3.4, there exists $N_1 > 0$ such that for any $n > N_1$,

$$v_n([y_{2k-1}, y_{2k}]) \leq 2(y_{2k} - y_{2k-1})e^{|\beta|},$$

for all $k \in [m]$. Since $v_n \xrightarrow{d} v$, there exists $N_2 > 0$ such that for any $n > N_2$,

$$|F_v(y_{2k}) - F_v(y_{2k-1}) - v_n([y_{2k-1}, y_{2k}])| < \frac{\epsilon}{2m},$$

for all $k \in [m]$. Let $n = \max(N_1, N_2) + 1$, we have

$$\begin{aligned} & \sum_{k=1}^m |F_v(y_{2k}) - F_v(y_{2k-1})| \\ & \leq \sum_{k=1}^m |F_v(y_{2k}) - F_v(y_{2k-1}) - v_n([y_{2k-1}, y_{2k}])| + \sum_{k=1}^m v_n([y_{2k-1}, y_{2k}]) \\ & < \frac{\epsilon}{2} + 2e^{|\beta|} \sum_{k=1}^m (y_{2k} - y_{2k-1}) \\ & < \frac{\epsilon}{2} + 2e^{|\beta|} \delta \\ & = \epsilon. \end{aligned}$$

□

Lemma 3.10. *In the context of Lemma 3.9, we have*

$$F_v(y) = \int_0^y u(a, t) dt,$$

for any $y \in [0, 1]$. Here $u(x, y)$ is defined in Theorem 1.2.

Proof. By Lemma 3.9, $F_v(y)$ is absolutely continuous. Hence $F_v(y)$ is differentiable almost everywhere, say $F'_v(y) = f(y)$ a.e. on $[0, 1]$, and moreover, we have $F_v(y) = \int_0^y f(t) dt$. Here we use the fact that the support of v is $[0, 1]$. Note that, by Lemma 3.4, for any $y \in (0, 1)$ such that $F'_v(y) = f(y)$, we have

$f(y) \geq e^{-|\beta|} > 0$. Then in order to show $f(y) = u(a, y)$ a.e., it suffices to show

$$\frac{f(y_2)}{f(y_1)} = \frac{u(a, y_2)}{u(a, y_1)}, \quad (27)$$

for any $y_1, y_2 \in A$, where $A := \{y \in (0, 1) : F'_v(y) = f(y)\}$. This is because, for any $y \in A$, we have

$$\frac{1}{f(y)} = \int_0^1 \frac{f(z)}{f(y)} dz = \int_A \frac{f(z)}{f(y)} dz = \int_A \frac{u(a, z)}{u(a, y)} dz = \int_0^1 \frac{u(a, z)}{u(a, y)} dz = \frac{1}{u(a, y)}.$$

Here we use the fact that the Lebesgue measure of A is 1 as well as Lemma 3.5 in the last equality. Next, since we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \frac{v((y_2, y_2 + \epsilon))}{v((y_1, y_1 + \epsilon))} &= \lim_{\epsilon \rightarrow 0^+} \frac{F_v(y_2 + \epsilon) - F_v(y_2)}{F_v(y_1 + \epsilon) - F_v(y_1)} \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{F_v(y_2 + \epsilon) - F_v(y_2)}{\epsilon} \bigg/ \frac{F_v(y_1 + \epsilon) - F_v(y_1)}{\epsilon} \\ &= \frac{f(y_2)}{f(y_1)}. \end{aligned}$$

Thus, to prove (27), it suffices to show that

$$\lim_{\epsilon \rightarrow 0^+} \left| \frac{v((y_2, y_2 + \epsilon))}{v((y_1, y_1 + \epsilon))} - \frac{u(a, y_2)}{u(a, y_1)} \right| = 0. \quad (28)$$

Next, inheriting the notation in (26), since $v_n \xrightarrow{d} v$ and $F_v(y)$ is continuous, we have

$$\lim_{n \rightarrow \infty} \left| \frac{v_n((y_2, y_2 + \epsilon))}{v_n((y_1, y_1 + \epsilon))} - \frac{u(a, y_2)}{u(a, y_1)} \right| = \left| \frac{v((y_2, y_2 + \epsilon))}{v((y_1, y_1 + \epsilon))} - \frac{u(a, y_2)}{u(a, y_1)} \right|. \quad (29)$$

Since $\{v_n\}$ is a subsequence of $\left\{ \mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (\cdot) \right) \right\}$, by Lemma 3.7, (28) follows from (29). \square

Proof of Theorem 1.2. Since the support of $\mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (\cdot) \right)$ is within $[0, 1]$, the sequence $\left\{ \mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \in (\cdot) \right) \right\}$ is tight. The claim follows from Lemma 3.9, Lemma 3.10 and Theorem 3.8. \square

4 Proof of the convergence of the empirical measure

Recall that, under the conditons in Theorem 1.3, we need to show the convergence of the empirical measure induced by $\{(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n})\}_{i \in [n]}$. Note that, by relabeling the indices, we have $\{(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n})\}_{i \in [n]} = \{(\frac{\pi^{-1}(i)}{n}, \frac{\tau(i)}{n})\}_{i \in [n]}$. Since π and τ are independent, for a give i , the x coordinate and y coordinate of $(\frac{\pi^{-1}(i)}{n}, \frac{\tau(i)}{n})$ are independent. We will exploit this property to establish the first and second moment estimates of the numbers of these points which fall inside an arbitrary rectangle. The following two lemmas play the key role in the proof of Theorem 1.3, and their will be presented in Section 5.

Lemma 4.1. *Suppose $A = [y_1, y_2] \subset [0, 1]$. For any $\beta \in \mathbb{R}$ and any sequence $\{q_n\}$ such that $q_n > 0$ and $\lim_{n \rightarrow \infty} n(1 - q_n) = \beta$,*

$$\lim_{n \rightarrow \infty} \max_{i \in [n]} \left| \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) - \int_{y_1}^{y_2} u \left(\frac{i}{n}, y, \beta \right) dy \right| = 0.$$

Lemma 4.2. *Suppose $A = [y_1, y_2] \subset [0, 1]$. Given $\beta \in \mathbb{R}$ and any sequence $\{q_n\}$ such that $q_n > 0$ and $\lim_{n \rightarrow \infty} n(1 - q_n) = \beta$, define*

$$\begin{aligned} \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) := \\ \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) - \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right). \end{aligned}$$

Then, we have

$$\lim_{n \rightarrow \infty} \max_{\substack{i \neq j \\ i, j \in [n]}} \left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right| = 0.$$

Lemma 4.1 states that under the same condition as in Theorem 1.2, the probability of $\pi(i)/n$ falling in an arbitrary interval converges to a constant uniformly for $i \in [n]$. Lemma 4.2 states that the covariance of $\mathbb{1}_A(\frac{\pi(i)}{n})$ and $\mathbb{1}_A(\frac{\pi(j)}{n})$ converges to 0 uniformly on all those pairs such that $i \neq j$. Recall that, in Section 2, we defined

$$L_\pi(R) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_R \left(\frac{i}{n}, \frac{\pi(i)}{n} \right), \quad \forall R \in \mathcal{B}_{[0,1] \times [0,1]}.$$

Similarly, we define

$$L_{\pi,\tau}(R) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_R\left(\frac{\pi(i)}{n}, \frac{\tau(i)}{n}\right), \quad \forall R \in \mathcal{B}_{[0,1] \times [0,1]}.$$

The lemmas above imply the following convergence for $L_{\pi,\tau}(R)$.

Lemma 4.3. *Under the same conditions as Theorem 1.3, for any $R = (x_1, x_2] \times (y_1, y_2] \subset [0, 1] \times [0, 1]$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left(\left| L_{\pi,\tau}(R) - \int_R \rho(x, y) dx dy \right| > \epsilon \right) = 0 \quad (30)$$

for any $\epsilon > 0$. Here $\rho(x, y)$ is the density function defined in Theorem 1.3.

Proof. Let $\bar{R} = [x_1, x_2] \times [y_1, y_2]$ be the closure of R . Since, for any vertical or horizontal line l and any $\pi, \tau \in S_n$, we have $L_{\pi,\tau}(l) \leq \frac{1}{n}$, it follows that

$$\left| L_{\pi,\tau}(R) - L_{\pi,\tau}(\bar{R}) \right| \leq \frac{2}{n}.$$

Then, given $\epsilon > 0$, for any $n > \frac{4}{\epsilon}$, by triangle inequality and the fact that $\int_R \rho(x, y) dx dy = \int_{\bar{R}} \rho(x, y) dx dy$, we get

$$\begin{aligned} & \left| L_{\pi,\tau}(R) - \int_R \rho(x, y) dx dy \right| > \epsilon \\ \Rightarrow & \left| L_{\pi,\tau}(\bar{R}) - \int_{\bar{R}} \rho(x, y) dx dy \right| > \frac{\epsilon}{2}. \end{aligned}$$

Hence, it suffices to show (30) for $R = [x_1, x_2] \times [y_1, y_2]$. In the remaining of the proof, let $R := [x_1, x_2] \times [y_1, y_2]$. We will show

$$\lim_{n \rightarrow \infty} \mathbb{E}_n(L_{\pi,\tau}(R)) = \int_R \rho(x, y) dx dy, \quad (31)$$

$$\lim_{n \rightarrow \infty} \text{Var}_n(L_{\pi,\tau}(R)) = 0. \quad (32)$$

Then, (30) follows from (31) and (32) by Chebyshev's inequality and triangle inequality.

Let $A = [x_1, x_2]$ and $B = [y_1, y_2]$. Define

$$\delta_n^{(i)} := \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n})) - \int_A u(x, \frac{i}{n}, \beta) dx,$$

$$\delta_n'^{(i)} := \mu_{n,q_n'}(\mathbb{1}_B(\frac{\tau(i)}{n})) - \int_B u(\frac{i}{n}, y, \gamma) dy,$$

$$\delta_n := \max_{i \in [n]}(|\delta_n^{(i)}|) \quad \text{and} \quad \delta_n' := \max_{i \in [n]}(|\delta_n'^{(i)}|).$$

Then, by Lemma 4.1 and the fact that $u(x, y, \beta) = u(y, x, \beta)$, for any $\epsilon > 0$, there exists $N_1 > 0$ such that, for any $n > N_1$,

$$\delta_n < \frac{\epsilon}{3} \quad \text{and} \quad \delta_n' < \frac{\epsilon}{3}.$$

Without loss of generality, assume $0 < \epsilon < 1$. Then, for any $n > N_1$ and any $i \in [n]$, we have

$$\begin{aligned} & \left| \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n}))\mu_{n,q_n'}(\mathbb{1}_B(\frac{\tau(i)}{n})) - \int_R u(x, \frac{i}{n}, \beta)u(\frac{i}{n}, y, \gamma) dx dy \right| \quad (33) \\ &= \left| \left(\delta_n^{(i)} + \int_A u(x, \frac{i}{n}, \beta) dx \right) \left(\delta_n'^{(i)} + \int_B u(\frac{i}{n}, y, \gamma) dy \right) \right. \\ & \quad \left. - \int_A u(x, \frac{i}{n}, \beta) dx \cdot \int_B u(\frac{i}{n}, y, \gamma) dy \right| \\ &\leq \left| \delta_n^{(i)} \right| \int_A u(x, \frac{i}{n}, \beta) dx + \left| \delta_n'^{(i)} \right| \int_B u(\frac{i}{n}, y, \gamma) dy + \left| \delta_n^{(i)} \delta_n'^{(i)} \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Here we use Lemma 3.5 in the last inequality. Hence, for any $n > N_1$,

$$\begin{aligned} & \left| \mathbb{E}_n(L_{\pi,\tau}(R)) - \frac{1}{n} \sum_{i=1}^n \int_R u(x, \frac{i}{n}, \beta)u(\frac{i}{n}, y, \gamma) dx dy \right| \quad (34) \\ &= \left| \frac{1}{n} \sum_{i=1}^n \mathbb{E}_n \left(\mathbb{1}_R(\frac{\pi(i)}{n}, \frac{\tau(i)}{n}) \right) - \frac{1}{n} \sum_{i=1}^n \int_R u(x, \frac{i}{n}, \beta)u(\frac{i}{n}, y, \gamma) dx dy \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n \left| \mathbb{E}_n \left(\mathbb{1}_A(\frac{\pi(i)}{n})\mathbb{1}_B(\frac{\tau(i)}{n}) \right) - \int_R u(x, \frac{i}{n}, \beta)u(\frac{i}{n}, y, \gamma) dx dy \right| \\ &= \frac{1}{n} \sum_{i=1}^n \left| \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n}))\mu_{n,q_n'}(\mathbb{1}_B(\frac{\tau(i)}{n})) - \int_R u(x, \frac{i}{n}, \beta)u(\frac{i}{n}, y, \gamma) dx dy \right| \\ &< \epsilon. \end{aligned}$$

Here the last equality follows from the fact that $(\pi, \tau) \sim \mu_{n,q_n} \times \mu_{n,q_n'}$ under \mathbb{P}_n , and the last inequality follows from (33).

Since $u(x, y, \beta)$ and $u(x, y, \gamma)$ are bounded on $[0, 1] \times [0, 1]$, by the definition of Riemann integral and the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \int_R u(x, \frac{i}{n}, \beta)u(\frac{i}{n}, y, \gamma) dx dy \quad (35)$$

$$\begin{aligned}
&= \int_R \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n u\left(x, \frac{i}{n}, \beta\right) u\left(\frac{i}{n}, y, \gamma\right) \right) dx dy \\
&= \int_R \left(\int_0^1 u(x, t, \beta) u(t, y, \gamma) dt \right) dx dy \\
&= \int_R \rho(x, y) dx dy.
\end{aligned}$$

Hence, (31) follows from (34) and (35).

To show (32), similarly, by Lemma 4.2, for any $\epsilon > 0$, there exists $N_2 > 0$ such that, for any $n > N_2$,

$$\begin{aligned}
&\max_{\substack{i \neq j \\ i, j \in [n]}} \left| \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) - \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right| < \frac{\epsilon}{4}, \\
&\max_{\substack{i \neq j \\ i, j \in [n]}} \left| \mu_{n, q'_n} \left(\mathbb{1}_B \left(\frac{\tau(i)}{n} \right) \mathbb{1}_B \left(\frac{\tau(j)}{n} \right) \right) - \mu_{n, q'_n} \left(\mathbb{1}_B \left(\frac{\tau(i)}{n} \right) \right) \mu_{n, q'_n} \left(\mathbb{1}_B \left(\frac{\tau(j)}{n} \right) \right) \right| < \frac{\epsilon}{4}.
\end{aligned}$$

Without loss of generality, assume $0 < \epsilon < 1$. Then, similar to (33), for any $n > N_2$ and any $1 \leq i < j \leq n$,

$$\begin{aligned}
&\left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_B \left(\frac{\tau(i)}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \mathbb{1}_B \left(\frac{\tau(j)}{n} \right) \right) \right| \tag{36} \\
&= \left| \mathbb{E}_n \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_B \left(\frac{\tau(i)}{n} \right) \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \mathbb{1}_B \left(\frac{\tau(j)}{n} \right) \right) \right. \\
&\quad \left. - \mathbb{E}_n \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_B \left(\frac{\tau(i)}{n} \right) \right) \mathbb{E}_n \left(\mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \mathbb{1}_B \left(\frac{\tau(j)}{n} \right) \right) \right| \\
&= \left| \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \mu_{n, q'_n} \left(\mathbb{1}_B \left(\frac{\tau(i)}{n} \right) \mathbb{1}_B \left(\frac{\tau(j)}{n} \right) \right) \right. \\
&\quad \left. - \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \mu_{n, q'_n} \left(\mathbb{1}_B \left(\frac{\tau(i)}{n} \right) \right) \mu_{n, q'_n} \left(\mathbb{1}_B \left(\frac{\tau(j)}{n} \right) \right) \right| \\
&< \frac{\epsilon}{2}.
\end{aligned}$$

Here the second equality follows from the fact that $(\pi, \tau) \sim \mu_{n, q_n} \times \mu_{n, q'_n}$ under \mathbb{P}_n , and the last inequality follows by triangle inequality. Specifically, if $0 \leq a_1, a_2, b_1, b_2 \leq 1$, $|a_1 - a_2| < \frac{\epsilon}{4}$ and $|b_1 - b_2| < \frac{\epsilon}{4}$, then we have

$$|a_1 b_1 - a_2 b_2| \leq |a_1 b_1 - a_2 b_1| + |a_2 b_1 - a_2 b_2| \leq |a_1 - a_2| + |b_1 - b_2| < \frac{\epsilon}{2}.$$

Here we choose

$$a_1 = \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right), \quad a_2 = \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right),$$

$$b_1 = \mu_{n,q'_n}(\mathbb{1}_B(\frac{\tau(i)}{n})\mathbb{1}_B(\frac{\tau(j)}{n})), \quad b_2 = \mu_{n,q'_n}(\mathbb{1}_B(\frac{\tau(i)}{n}))\mu_{n,q'_n}(\mathbb{1}_B(\frac{\tau(j)}{n})).$$

Thus, for any $n > \max(N_2, \frac{1}{\epsilon})$,

$$\begin{aligned} & \text{Var}_n(L_{\pi,\tau}(R)) \\ &= \text{Var}_n\left(\frac{1}{n} \sum_{i=1}^n \mathbb{1}_A\left(\frac{\pi(i)}{n}\right) \mathbb{1}_B\left(\frac{\tau(i)}{n}\right)\right) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}_n(\mathbb{1}_A\left(\frac{\pi(i)}{n}\right) \mathbb{1}_B\left(\frac{\tau(i)}{n}\right)) \\ & \quad + \frac{1}{n^2} \sum_{\substack{i \neq j \\ i,j \in [n]}} \text{Cov}_n(\mathbb{1}_A\left(\frac{\pi(i)}{n}\right) \mathbb{1}_B\left(\frac{\tau(i)}{n}\right), \mathbb{1}_A\left(\frac{\pi(j)}{n}\right) \mathbb{1}_B\left(\frac{\tau(j)}{n}\right)) \\ &< \frac{1}{n^2} \cdot \frac{n}{4} + \frac{n(n-1)}{n^2} \cdot \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

The first inequality follows by (36) and the fact that the variance of any indicator function is no greater than $\frac{1}{4}$. \square

Now we are in the position to prove Theorem 1.3

Proof of Theorem 1.3. First of all, we make the following claim:

Claim: To prove Theorem 1.3, it suffices to show the case when $f(x, y) = \mathbb{1}_R(x, y)$, for any $R = (x_1, x_2] \times (y_1, y_2] \subset [0, 1] \times [0, 1]$. This is because for any continuous function $f(x, y)$ and any $\epsilon > 0$, we can find a simple function $s(x, y)$ on $(0, 1] \times (0, 1]$ such that

$$|f(x, y) - s(x, y)| < \frac{\epsilon}{3} \quad \forall (x, y) \in (0, 1] \times (0, 1],$$

where $s(x, y)$ is of the form

$$s(x, y) = \sum_{j=1}^m a_j \mathbb{1}_{R_j}(x, y),$$

with $R_j = \left(x_1^{(j)}, x_2^{(j)}\right] \times \left(y_1^{(j)}, y_2^{(j)}\right] \subset (0, 1] \times (0, 1]$ and $\{R_j\}_{j=1}^m$ is a partition of $(0, 1] \times (0, 1]$. Hence, we have

$$\left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n}\right) - \frac{1}{n} \sum_{i=1}^n s\left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n}\right) \right| < \frac{\epsilon}{3}, \quad (37)$$

and

$$\begin{aligned}
& \left| \int_0^1 \int_0^1 s(x, y) \rho(x, y) dx dy - \int_0^1 \int_0^1 f(x, y) \rho(x, y) dx dy \right| \\
& \leq \int_0^1 \int_0^1 |s(x, y) - f(x, y)| \rho(x, y) dx dy \\
& < \frac{\epsilon}{3}.
\end{aligned} \tag{38}$$

Here we use the fact that, by Lemma 3.5,

$$\int_0^1 \int_0^1 \rho(x, y) dx dy = 1.$$

Thus, by (37), (38) and triangle inequality, we have

$$\begin{aligned}
& \left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n}\right) - \int_0^1 \int_0^1 f(x, y) \rho(x, y) dx dy \right| > \epsilon \\
& \Rightarrow \left| \frac{1}{n} \sum_{i=1}^n s\left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n}\right) - \int_0^1 \int_0^1 s(x, y) \rho(x, y) dx dy \right| > \frac{\epsilon}{3}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
& \mathbb{P}_n \left(\left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n}\right) - \int_0^1 \int_0^1 f(x, y) \rho(x, y) dx dy \right| > \epsilon \right) \\
& \leq \mathbb{P}_n \left(\left| \frac{1}{n} \sum_{i=1}^n s\left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n}\right) - \int_0^1 \int_0^1 s(x, y) \rho(x, y) dx dy \right| > \frac{\epsilon}{3} \right) \\
& \leq \sum_{j=1}^m \mathbb{P}_n \left(\left| \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{R_j}\left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n}\right) - \int_{R_j} \rho(x, y) dx dy \right| > \frac{\epsilon}{3m|a_j|} \right).
\end{aligned}$$

Here the last inequality follows by the union bound. Therefore, to prove Theorem 1.3, it suffices to show the case when $f(x, y) = \mathbb{1}_R(x, y)$, with $R = (x_1, x_2] \times (y_1, y_2]$. In other words, we need to show that, for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left(\left| L_{\tau \circ \pi}(R) - \int_R \rho(x, y) dx dy \right| > \epsilon \right) = 0. \tag{39}$$

Here, as defined in Section 2,

$$L_{\tau \circ \pi}(R) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_R \left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n} \right).$$

Then, for any $\pi, \tau \in S_n$, we have

$$\{(i, \tau \circ \pi(i)) : i \in [n]\} = \{(\pi^{-1}(\pi(i)), \tau(\pi(i))) : i \in [n]\}$$

$$= \{(\pi^{-1}(i), \tau(i)) : i \in [n]\}.$$

The last equality follows since $\{\pi(i)\}_{i \in [n]} = \{i\}_{i \in [n]}$. Thus, it follows that

$$L_{\tau \circ \pi}(R) = L_{\pi^{-1}, \tau}(R), \quad \forall R \in \mathcal{B}_{[0,1] \times [0,1]}.$$

If $(\pi, \tau) \sim \mu_{n,q} \times \mu_{n,q'}$, by Proposition 2.4, $(\pi^{-1}, \tau) \sim \mu_{n,q} \times \mu_{n,q'}$. Thus, given $(\pi, \tau) \sim \mathbb{P}_n$, we have

$$L_{\tau \circ \pi}(R) = L_{\pi^{-1}, \tau}(R) \stackrel{d}{=} L_{\pi, \tau}(R).$$

That is $L_{\tau \circ \pi}(R)$ and $L_{\pi, \tau}(R)$ have the same distribution when $(\pi, \tau) \sim \mathbb{P}_n$. Therefore, (39) follows by Lemma 4.3. □

5 Proof of Lemma 4.1 and Lemma 4.2

We now complete the proofs of Lemmas 4.1 and 4.2.

Definition 5.1. For any $\pi \in S_n$ and any $1 \leq j < k \leq n$, let $\pi([j, k])$ denote the vector $(\pi(j), \pi(j+1), \dots, \pi(k))$. Let $\pi_{[j, k]}$ denote the permutation in S_{k-j+1} induced by $\pi([j, k])$, i. e.

$$\pi_{[j, k]}(i) = \sum_{s=j}^k \mathbf{1}_{\{\pi(s) \leq \pi(j+i-1)\}}, \quad \forall i \in [k-j+1].$$

We make use of the following property of the Mallows distribution (see e.g. Lemma 2.5 and Lemma 2.6 in [2]):

Proposition 5.2. Given $\pi \sim \mu_{n,q}$, for any $1 < k < n$, we have $\pi_{[1, k]} \sim \mu_{k,q}$, $\pi_{[k+1, n]} \sim \mu_{n-k,q}$ and $\pi_{[1, k]}, \pi_{[k+1, n]}$ are independent.

Lemma 5.3. For any $0 \leq a < b \leq 1$ and $y \in [0, 1]$, we have the following identity

$$\int_0^y u(a, t, \beta) dt = \int_0^{y'} u\left(\frac{a}{b}, t, b\beta\right) dt, \quad \forall \beta \in \mathbb{R}.$$

Here,

$$y' := \frac{1}{b} u_\beta([0, b] \times [0, y]) = \frac{1}{b} \int_0^b \int_0^y u(x, t, \beta) dt dx$$

and $u(x, y, \beta)$ is defined in Theorem 1.2.

We make some preparation before proving Lemma 5.3. Given $a, b \in [0, 1]$, choose two sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \in [n]$, $b_n \in [n]$ and $\lim_{n \rightarrow \infty} \frac{a_n}{n} = a$, $\lim_{n \rightarrow \infty} \frac{b_n}{n} = b$. Moreover, for any $\beta \in \mathbb{R}$, choose a sequence $\{q_n\}$ with $q_n > 0$ such that $\lim_{n \rightarrow \infty} n(1 - q_n) = \beta$. By Theorem 1.2, we have

$$\lim_{n \rightarrow \infty} \mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \leq y \right) = \int_0^y u(a, t, \beta) dt. \quad (40)$$

We will show that

$$\lim_{n \rightarrow \infty} \mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \leq y \right) = \int_0^y u\left(\frac{a}{b}, t, b\beta\right) dt. \quad (41)$$

Lemma 5.3 follows from (40) and (41). First, regarding $\{a_n\}$ and $\{b_n\}$ as fixed sequences, y as a fixed number, we make the following definitions,

$$R_0 := [0, b] \times [0, y], \quad R_1 := [b, 1] \times [0, y],$$

$$K_n := \{(v_1, v_2, \dots, v_{n-b_n+1}) : v_i \in [n] \text{ and } i \neq j \Rightarrow v_i \neq v_j\},$$

$$f_n(v) := |\{v_i \in v : v_i \leq ny\}| \quad \text{for } v \in K_n,$$

$$G_n(\lambda) := \left\{ v \in K_n : \left| \frac{1}{n} f_n(v) - u_\beta(R_1) \right| < \lambda \right\}.$$

Here K_n consists of all possible values $\pi([b_n, n])$ can take when $\pi \in S_n$. And $f_n(\pi([b_n, n]))$ denotes the number of points $\left(\frac{i}{n}, \frac{\pi(i)}{n}\right)$ inside the rectangle $[\frac{b_n}{n}, 1] \times [0, y]$.

Next we show that, for any $\lambda > 0$,

$$\lim_{n \rightarrow \infty} \mu_{n, q_n} (\pi([b_n, n]) \notin G_n(\lambda)) = 0 \quad (42)$$

Proof of (42). First, since the difference between $[\frac{b_n}{n}, 1] \times [0, y]$ and R_1 is a rectangle with width $|\frac{b_n}{n} - b|$, it follows that

$$\begin{aligned} & |f_n(\pi([b_n, n])) - nL_\pi(R_1)| \\ &= \left| \left| \left\{ i : \left(\frac{i}{n}, \frac{\pi(i)}{n} \right) \in [\frac{b_n}{n}, 1] \times [0, y] \right\} \right| - \left| \left\{ i : \left(\frac{i}{n}, \frac{\pi(i)}{n} \right) \in R_1 \right\} \right| \right| \\ &\leq |b_n - nb| + 1. \end{aligned}$$

Thus, for any $\lambda > 0$, there exists a $N > 0$ such that for all $n > N$,

$$\left| \frac{1}{n} f_n(\pi([b_n, n])) - L_\pi(R_1) \right| \leq \left| \frac{b_n}{n} - b \right| + \frac{1}{n} < \frac{\lambda}{2}.$$

Here we use the fact that $\lim_{n \rightarrow \infty} \frac{b_n}{n} = b$. Hence, for any $n > N$, we have

$$\begin{aligned} & \left| \frac{1}{n} f_n(\pi([b_n, n])) - u_\beta(R_1) \right| \geq \lambda \\ \Rightarrow & \left| \frac{1}{n} f_n(\pi([b_n, n])) - L_\pi(R_1) \right| + |L_\pi(R_1) - u_\beta(R_1)| \geq \lambda \\ \Rightarrow & |L_\pi(R_1) - u_\beta(R_1)| > \frac{\lambda}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} & \mu_{n,q_n}(\pi([b_n, n]) \notin G_n(\lambda)) \\ &= \mu_{n,q_n} \left(\left| \frac{1}{n} f_n(\pi([b_n, n])) - u_\beta(R_1) \right| \geq \lambda \right) \\ &\leq \mu_{n,q_n} \left(|L_\pi(R_1) - u_\beta(R_1)| > \frac{\lambda}{2} \right). \end{aligned}$$

(42) follows from the above inequality and Lemma 3.2. \square

Next we show that, for any $\epsilon > 0$, we can choose a sufficiently small λ and $N > 0$ such that for all $n > N$ and any $v \in G_n(\lambda)$,

$$\left| \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \leq y \mid \pi([b_n, n]) = v \right) - \int_0^{y'} u\left(\frac{a}{b}, t, b\beta\right) dt \right| < \frac{\epsilon}{3}. \quad (43)$$

Proof of (43). Assume n is sufficiently large such that $a_n < b_n$. For any $v \in G_n(\lambda)$, here the value of λ is to be determined, we have

$$\begin{aligned} & \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \leq y \mid \pi([b_n, n]) = v \right) \\ &= \mu_{n,q_n} (\pi(a_n) \leq ny \mid \pi([b_n, n]) = v) \\ &= \mu_{n,q_n} (\pi_{[1,b_n-1]}(a_n) \leq \lfloor ny \rfloor - f_n(v) \mid \pi([b_n, n]) = v) \\ &= \mu_{b_n-1,q_n} (\tau(a_n) \leq \lfloor ny \rfloor - f_n(v)) \\ &= \mu_{b_n-1,q_n} \left(\frac{\tau(a_n)}{b_n-1} \leq \frac{1}{b_n-1} (\lfloor ny \rfloor - f_n(v)) \right) \\ &= \mu_{b_n-1,q_n} \left(\frac{\tau(a_n)}{b_n-1} \leq \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - \frac{f_n(v)}{n} \right) \right). \end{aligned} \quad (44)$$

The second equality follows since, conditioned on $\pi([b_n, n]) = v$, we have

$$\{\pi \in S_n : \pi(a_n) \leq ny\} = \{\pi \in S_n : \pi_{[1,b_n-1]}(a_n) \leq \lfloor ny \rfloor - f_n(v)\}.$$

Note that $\lfloor ny \rfloor - f_n(v)$ is the number of $i \leq ny$ which is not in v . The third equality is due to Proposition 5.2 with $\tau \sim \mu_{b_n-1, q_n}$. Next, by the following facts,

$$\lim_{n \rightarrow \infty} (b_n - 1)(1 - q_n) = \lim_{n \rightarrow \infty} \frac{b_n - 1}{n} \cdot \lim_{n \rightarrow \infty} n(1 - q_n) = b\beta, \quad (45)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n - 1} &= \lim_{n \rightarrow \infty} \frac{a_n}{n} \cdot \lim_{n \rightarrow \infty} \frac{n}{b_n - 1} = \frac{a}{b}, \\ \lim_{n \rightarrow \infty} \frac{n}{b_n - 1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right) &= \frac{1}{b}(y - u_\beta(R_1)) = \frac{1}{b} u_\beta(R_0) = y', \end{aligned} \quad (46)$$

and Theorem 1.2, we have

$$\lim_{n \rightarrow \infty} \mu_{b_n-1, q_n} \left(\frac{\tau(a_n)}{b_n-1} \leq \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right) \right) = \int_0^{y'} u\left(\frac{a}{b}, t, b\beta\right) dt.$$

Hence, there exists $N_1 > 0$ such that for any $n > N_1$,

$$\left| \mu_{b_n-1, q_n} \left(\frac{\tau(a_n)}{b_n-1} \leq \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right) \right) - \int_0^{y'} u\left(\frac{a}{b}, t, b\beta\right) dt \right| < \frac{\epsilon}{6}. \quad (47)$$

By (46), there exists $N_2 > 0$ such that for all $n > N_2$,

$$\frac{n}{b_n-1} < \frac{2}{b} \quad \text{and} \quad \left| \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right) - y' \right| < \lambda. \quad (48)$$

Hence, for any $n > N_2$ and any $v \in G_n(\lambda)$, we have

$$\begin{aligned} & \left| \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - \frac{f_n(v)}{n} \right) - y' \right| \\ & \leq \left| \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - \frac{f_n(v)}{n} \right) - \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right) \right| \\ & \quad + \left| \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right) - y' \right| \\ & < \left(\frac{2}{b} + 1 \right) \lambda. \end{aligned} \quad (49)$$

Let $C := \frac{2}{b} + 1$. Since, by (48) and (49), both $\frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - \frac{f_n(v)}{n} \right)$ and $\frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right)$ are in the interval $(y' - C\lambda, y' + C\lambda)$, it follows that, for any $n > N_2$ and any $v \in G_n(\lambda)$,

$$\begin{aligned} & \left| \mu_{b_n-1, q_n} \left(\frac{\tau(a_n)}{b_n-1} \leq \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - \frac{f_n(v)}{n} \right) \right) \right. \\ & \quad \left. - \mu_{b_n-1, q_n} \left(\frac{\tau(a_n)}{b_n-1} \leq \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - u_\beta(R_1) \right) \right) \right| \end{aligned} \quad (50)$$

$$< \mu_{b_n-1, q_n} \left(\frac{\tau(a_n)}{b_n-1} \in (y' - C\lambda, y' + C\lambda) \right).$$

By (45) and Lemma 3.4, there exists $N_3 > 0$ such that for all $n > N_3$,

$$\mu_{b_n-1, q_n} \left(\frac{\tau(a_n)}{b_n-1} \in (y' - C\lambda, y' + C\lambda) \right) < 4C\lambda e^{b|\beta|}. \quad (51)$$

Therefore, we can fix $\lambda = \frac{\epsilon}{24C} e^{-b|\beta|}$ in the first place. Then, by (47), (50) and (51), for any $n > \max(N_1, N_2, N_3)$ and any $v \in G_n(\lambda)$,

$$\begin{aligned} & \left| \mu_{b_n-1, q_n} \left(\frac{\tau(a_n)}{b_n-1} \leq \frac{n}{b_n-1} \left(\frac{\lfloor ny \rfloor}{n} - \frac{f_n(v)}{n} \right) \right) - \int_0^{y'} u \left(\frac{a}{b}, t, b\beta \right) dt \right| \\ & < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}. \end{aligned} \quad (52)$$

(43) follows by (44) and (52). \square

Now we are in the position to prove (41), which completes the proof of Lemma 5.3.

Proof of (41). For simplicity, let $I := \int_0^{y'} u \left(\frac{a}{b}, t, b\beta \right) dt$. Since

$$y' = \frac{1}{b} u_\beta(R_0) \leq \frac{1}{b} u_\beta([0, b] \times [0, 1]) = 1,$$

we have

$$I = \int_0^{y'} u \left(\frac{a}{b}, t, b\beta \right) dt \leq \int_0^1 u \left(\frac{a}{b}, t, b\beta \right) dt = 1.$$

Then, given $\epsilon > 0$, fix the value of λ such that (43) holds for any $n > N_1$ and any $v \in G_n(\lambda)$. By (42), there exists $N_2 > 0$ such that for any $n > N_2$,

$$\mu_{n, q_n}(\pi([b_n, n]) \notin G_n(\lambda)) < \frac{\epsilon}{3}. \quad (53)$$

Then, for any $n > \max(N_1, N_2)$,

$$\begin{aligned} & \left| \mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \leq y \right) - I \right| \\ &= \left| \sum_{v \in K_n} \mu_{n, q_n} \left(\frac{\pi(a_n)}{n} \leq y \mid \pi([b_n, n]) = v \right) \cdot \mu_{n, q_n}(\pi([b_n, n]) = v) \right. \\ & \quad \left. - \sum_{v \in K_n} I \cdot \mu_{n, q_n}(\pi([b_n, n]) = v) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{v \in G_n(\lambda)} \left| \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \leq y \mid \pi([b_n, n]) = v \right) - I \right| \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\
&\quad + \sum_{v \notin G_n(\lambda)} \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \leq y \mid \pi([b_n, n]) = v \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\
&\quad + \sum_{v \notin G_n(\lambda)} I \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\
&\leq \frac{\epsilon}{3} \cdot \sum_{v \in G_n(\lambda)} \mu_{n,q_n} (\pi([b_n, n]) = v) + 2 \cdot \sum_{v \notin G_n(\lambda)} \mu_{n,q_n} (\pi([b_n, n]) = v) \\
&< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.
\end{aligned}$$

Here we use (43) and (53) in the second to last inequality. \square

Lemma 5.4. *For any $0 \leq a < b \leq 1$ and any $\beta \in \mathbb{R}$, suppose we have sequences $\{a_n\}$, $\{b_n\}$ and $\{q_n\}$ such that $a_n \in [n]$, $b_n \in [n]$, $q_n > 0$ and*

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = a, \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = b, \quad \lim_{n \rightarrow \infty} n(1 - q_n) = \beta.$$

Then, for any $A = [y_1, y_2] \subset [0, 1]$ and $B = [y_3, y_4] \subset [0, 1]$,

$$\lim_{n \rightarrow \infty} \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n)}{n} \right) \mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) - \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n)}{n} \right) \right) \mu_{n,q_n} \left(\mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) = 0.$$

Proof. The proof is similar to the proof of Lemma 5.3, and we inherit those definitions in the previous proof. First of all, since

$$\begin{aligned}
&\mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n)}{n} \right) \mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) \\
&= \mu_{n,q_n} \left(\mathbb{1}_{[0,y_2]} \left(\frac{\pi(a_n)}{n} \right) \mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) - \mu_{n,q_n} \left(\mathbb{1}_{[0,y_1]} \left(\frac{\pi(a_n)}{n} \right) \mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right)
\end{aligned}$$

and

$$\mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n)}{n} \right) \right) = \mu_{n,q_n} \left(\mathbb{1}_{[0,y_2]} \left(\frac{\pi(a_n)}{n} \right) \right) - \mu_{n,q_n} \left(\mathbb{1}_{[0,y_1]} \left(\frac{\pi(a_n)}{n} \right) \right),$$

it suffices to show the cases when the interval A is of the form $[0, y]$ or $[0, y)$ for any $y \in [0, 1]$. Moreover, by Theorem 1.2, we have

$$\lim_{n \rightarrow \infty} \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} = y \right) = 0, \quad \forall y \in [0, 1].$$

Hence, it suffices to show the case when $A = [0, y]$, for any $y \in [0, 1]$. By Lemma 5.3, define

$$I := \int_0^y u(a, t, \beta) dt = \int_0^{y'} u\left(\frac{a}{b}, t, b\beta\right) dt.$$

By Theorem 1.2, we have

$$\lim_{n \rightarrow \infty} \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n)}{n} \right) \right) = \int_0^y u(a, t, \beta) dt = I.$$

Hence it suffices to show the following,

$$\lim_{n \rightarrow \infty} \mu_{n,q_n} \left(\mathbb{1} \left(\frac{\pi(a_n)}{n} \leq y \right) \mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) - \mu_{n,q_n} \left(\mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) \cdot I = 0, \quad (54)$$

for any $y \in [0, 1]$. Given $\epsilon > 0$, by (43), there exists $\lambda > 0$ and $N_1 > 0$ such that for any $n > N_1$ and any $v \in G_n(\lambda)$,

$$\left| \mu_{n,q_n} \left(\frac{\pi(a_n)}{n} \leq y \mid \pi([b_n, n]) = v \right) - I \right| < \frac{\epsilon}{3}. \quad (55)$$

By (42), there exists $N_2 > 0$ such that for any $n > N_2$,

$$\mu_{n,q_n} (\pi([b_n, n]) \notin G_n(\lambda)) < \frac{\epsilon}{3}. \quad (56)$$

Moreover, by conditioning on the value of $\pi([b_n, n])$, we have

$$\begin{aligned} & \mu_{n,q_n} \left(\mathbb{1} \left(\frac{\pi(a_n)}{n} \leq y \right) \mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) \\ &= \sum_{v \in K_n} \mu_{n,q_n} \left(\mathbb{1} \left(\frac{\pi(a_n)}{n} \leq y \right) \mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \mid \pi([b_n, n]) = v \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\ &= \sum_{v \in K_n} \mu_{n,q_n} \left(\mathbb{1} \left(\frac{\pi(a_n)}{n} \leq y \right) \mid \pi([b_n, n]) = v \right) \cdot \mathbb{1}_B \left(\frac{v_1}{n} \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\ &= \sum_{v \in G_n(\lambda)} \mu_{n,q_n} \left(\mathbb{1} \left(\frac{\pi(a_n)}{n} \leq y \right) \mid \pi([b_n, n]) = v \right) \cdot \mathbb{1}_B \left(\frac{v_1}{n} \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\ &+ \sum_{v \notin G_n(\lambda)} \mu_{n,q_n} \left(\mathbb{1} \left(\frac{\pi(a_n)}{n} \leq y \right) \mid \pi([b_n, n]) = v \right) \cdot \mathbb{1}_B \left(\frac{v_1}{n} \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \end{aligned}$$

and

$$\begin{aligned} & \mu_{n,q_n} \left(\mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \right) \\ &= \sum_{v \in K_n} \mu_{n,q_n} \left(\mathbb{1}_B \left(\frac{\pi(b_n)}{n} \right) \mid \pi([b_n, n]) = v \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\ &= \sum_{v \in K_n} \mathbb{1}_B \left(\frac{v_1}{n} \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \\ &= \sum_{v \in G_n(\lambda)} \mathbb{1}_B \left(\frac{v_1}{n} \right) \cdot \mu_{n,q_n} (\pi([b_n, n]) = v) \end{aligned}$$

$$+ \sum_{v \notin G_n(\lambda)} \mathbb{1}_B\left(\frac{v_1}{n}\right) \cdot \mu_{n,q_n}(\pi([b_n, n]) = v).$$

Here v_1 denotes the first entry of vector v . Hence, for any $n > \max(N_1, N_2)$, we have

$$\begin{aligned} & \left| \mu_{n,q_n} \left(\mathbb{1}\left(\frac{\pi(a_n)}{n} \leq y\right) \mathbb{1}_B\left(\frac{\pi(b_n)}{n}\right) \right) - \mu_{n,q_n} \left(\mathbb{1}_B\left(\frac{\pi(b_n)}{n}\right) \right) \cdot I \right| \\ & \leq \sum_{v \in G_n(\lambda)} \left| \mu_{n,q_n} \left(\mathbb{1}\left(\frac{\pi(a_n)}{n} \leq y\right) \mid \pi([b_n, n]) = v \right) - I \right| \cdot \mu_{n,q_n}(\pi([b_n, n]) = v) \\ & \quad + 2 \sum_{v \notin G_n(\lambda)} \mu_{n,q_n}(\pi([b_n, n]) = v) \\ & \leq \frac{\epsilon}{3} \sum_{v \in G_n(\lambda)} \mu_{n,q_n}(\pi([b_n, n]) = v) + 2 \sum_{v \notin G_n(\lambda)} \mu_{n,q_n}(\pi([b_n, n]) = v) \\ & < \frac{\epsilon}{3} + 2 \cdot \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

The first inequality follows from triangle inequality and the fact that,

$$\mu_{n,q_n} \left(\mathbb{1}\left(\frac{\pi(a_n)}{n} \leq y\right) \mid \pi([b_n, n]) = v \right) \leq 1, \quad \mathbb{1}_B\left(\frac{v_1}{n}\right) \leq 1, \text{ and } I \leq 1.$$

The last two inequalities follow from (55) and (56) respectively. \square

Before we start to prove Lemma 4.1 and Lemma 4.2, we briefly introduce the following facts:

Lemma 5.5. *For any $s, t, i \in [n]$,*

$$\min(q^d, q^{-d}) \leq \frac{\mu_{n,q}(\pi(s) = i)}{\mu_{n,q}(\pi(t) = i)} \leq \max(q^d, q^{-d}),$$

where $d = |s - t|$.

Lemma 5.6. *For any $s, t, w, i, j \in [n]$ such that either $w < \min(s, t)$ or $w > \max(s, t)$,*

$$\min(q^d, q^{-d}) \leq \frac{\mu_{n,q}(\{\pi \in S_n : \pi(s) = i \text{ and } \pi(w) = j\})}{\mu_{n,q}(\{\pi \in S_n : \pi(t) = i \text{ and } \pi(w) = j\})} \leq \max(q^d, q^{-d}),$$

where $d = |s - t|$.

These two lemmas follow from similar argument as in the proof of Lemma 3.3. We omit their proofs. From these two lemmas, we can show the following,

Lemma 5.7. *For any $A \subset [0, 1]$ and any $s, t \in [n]$,*

$$\left| \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(s)}{n} \right) \right) - \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(t)}{n} \right) \right) \right| \leq M,$$

where $M = \max(|1 - q^d|, |1 - q^{-d}|)$ and $d = |s - t|$.

Lemma 5.8. *For any $A, B \subset [0, 1]$ and any $s, t, w \in [n]$ such that either $w < \min(s, t)$ or $w > \max(s, t)$,*

$$\left| \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(s)}{n} \right) \mathbb{1}_B \left(\frac{\pi(w)}{n} \right) \right) - \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(t)}{n} \right) \mathbb{1}_B \left(\frac{\pi(w)}{n} \right) \right) \right| \leq M,$$

where $M = \max(|1 - q^d|, |1 - q^{-d}|)$ and $d = |s - t|$.

Here we only deduce Lemma 5.7 from Lemma 5.5. Lemma 5.8 follows from Lemma 5.6 by the similar argument.

Proof of Lemma 5.7. Without loss of generality, assume $0 < q < 1$. By Lemma 5.5, for any $i \in [n]$, we have

$$q^d \leq \frac{\mu_{n,q}(\pi(s) = i)}{\mu_{n,q}(\pi(t) = i)} \leq q^{-d}.$$

Hence

$$q^d \sum_{\{i: \frac{i}{n} \in A\}} \mu_{n,q}(\pi(t) = i) \leq \sum_{\{i: \frac{i}{n} \in A\}} \mu_{n,q}(\pi(s) = i) \leq q^{-d} \sum_{\{i: \frac{i}{n} \in A\}} \mu_{n,q}(\pi(t) = i).$$

Thus

$$\mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(t)}{n} \right) \right) \cdot q^d \leq \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(s)}{n} \right) \right) \leq \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(t)}{n} \right) \right) \cdot q^{-d}.$$

Therefore

$$\begin{aligned} & \left| \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(s)}{n} \right) \right) - \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(t)}{n} \right) \right) \right| \\ & \leq \mu_{n,q} \left(\mathbb{1}_A \left(\frac{\pi(t)}{n} \right) \right) \max(q^{-d} - 1, 1 - q^d) \\ & \leq \max(q^{-d} - 1, 1 - q^d). \end{aligned}$$

□

Proof of Lemma 4.2. Let m be a positive integer whose value is to be determined. Define the following $m + 1$ sequences $\{a_n^{(k)}\}$, $0 \leq k \leq m$, as follows,

$$a_n^{(k)} := \begin{cases} 1, & \text{if } k = 0; \\ \lceil \frac{kn}{m} \rceil, & \text{if } 1 \leq k \leq m. \end{cases} \quad (57)$$

Then, for any $0 \leq k \leq m$, we have $\lim_{n \rightarrow \infty} \frac{a_n^{(k)}}{n} = \frac{k}{m}$. Also, for any $0 \leq k \leq m - 1$ and $n > m$ we have

$$1 \leq a_n^{(k+1)} - a_n^{(k)} \leq \frac{n}{m} + 1.$$

Then, for any $n > m$ and any $i, j \in [n]$ with $i < j$, there exist unique k and l such that

$$i \in [a_n^{(k)}, a_n^{(k+1)}], \quad \text{and} \quad j \in (a_n^{(l-1)}, a_n^{(l)}]. \quad (58)$$

Clearly, we have

$$k < l, \quad |i - a_n^{(k)}| \leq \frac{n}{m} \quad \text{and} \quad |j - a_n^{(l)}| \leq \frac{n}{m}. \quad (59)$$

Then, given $\epsilon > 0$, fix a sufficiently large m in the first place such that,

$$\left| e^{\frac{\beta}{m}} - 1 \right| < \frac{\epsilon}{12}, \quad \left| e^{-\frac{\beta}{m}} - 1 \right| < \frac{\epsilon}{12}.$$

Next, since $\lim_{n \rightarrow \infty} q_n^n = e^{-\beta}$, there exists $N_1 > 0$ such that for any $n > N_1$,

$$\left| e^{\frac{\beta}{m}} - q_n^{-\frac{n}{m}} \right| < \frac{\epsilon}{12}, \quad \left| e^{-\frac{\beta}{m}} - q_n^{\frac{n}{m}} \right| < \frac{\epsilon}{12}.$$

Then, by triangle inequality, for any $n > N_1$,

$$\max \left(\left| 1 - q_n^{\frac{n}{m}} \right|, \left| 1 - q_n^{-\frac{n}{m}} \right| \right) < \frac{\epsilon}{6}. \quad (60)$$

For the simplicity of notation, define

$$\begin{aligned} U &:= \left| \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n})) \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(j)}{n})) - \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(a_n^{(k)})}{n})) \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(j)}{n})) \right|, \\ V &:= \left| \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n}) \mathbb{1}_A(\frac{\pi(j)}{n})) - \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(a_n^{(k)})}{n}) \mathbb{1}_A(\frac{\pi(j)}{n})) \right|, \\ W &:= \left| \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(a_n^{(k)})}{n}) \mathbb{1}_A(\frac{\pi(j)}{n})) - \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(a_n^{(k)})}{n})) \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(j)}{n})) \right| \end{aligned}$$

$$= \left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right|.$$

Then, by (59), (60), Lemma 5.7 and Lemma 5.8, for any $n > \max(m, N_1)$ and any $0 \leq i < j \leq n$ with corresponding k, l defined in (58), we have

$$\begin{aligned} U &= \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \cdot \left| \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) - \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right) \right) \right| \\ &\leq \left| \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) - \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right) \right) \right| \\ &\leq \max \left(\left| 1 - q_n^{\frac{n}{m}} \right|, \left| 1 - q_n^{-\frac{n}{m}} \right| \right) \\ &< \frac{\epsilon}{6}, \\ V &\leq \max \left(\left| 1 - q_n^{\frac{n}{m}} \right|, \left| 1 - q_n^{-\frac{n}{m}} \right| \right) < \frac{\epsilon}{6}. \end{aligned}$$

Whence, again, by triangle inequality, for any $n > \max(m, N_1)$,

$$\begin{aligned} &\left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right| \tag{61} \\ &= \left| \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) - \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) \mu_{n,q_n} \left(\mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right| \\ &< U + V + W \\ &< \left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right| + \frac{\epsilon}{3}. \end{aligned}$$

By the same argument, it follows that for any $n > \max(m, N_1)$,

$$\begin{aligned} &\left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right| \tag{62} \\ &< \left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right), \mathbb{1}_A \left(\frac{\pi(a_n^{(l)})}{n} \right) \right) \right| + \frac{\epsilon}{3}. \end{aligned}$$

Combining (61) and (62), for any $n > \max(m, N_1)$ and any $0 \leq i < j \leq n$ with corresponding k, l defined in (58), we have

$$\begin{aligned} &\left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right), \mathbb{1}_A \left(\frac{\pi(j)}{n} \right) \right) \right| \\ &< \left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right), \mathbb{1}_A \left(\frac{\pi(a_n^{(l)})}{n} \right) \right) \right| + \frac{2\epsilon}{3}. \end{aligned}$$

Moreover, since m is fixed, by Lemma 5.4, there exists $N_2 > 0$ such that, for any $n > N_2$ and any $0 \leq k < l \leq m$, we have

$$\left| \text{Cov}_n \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right), \mathbb{1}_A \left(\frac{\pi(a_n^{(l)})}{n} \right) \right) \right| < \frac{\epsilon}{3}.$$

Thus, for $n > \max(m, N_1, N_2)$ and any $0 \leq i < j \leq n$,

$$\left| \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n})\mathbb{1}_A(\frac{\pi(j)}{n})) - \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n}))\mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(j)}{n})) \right| < \epsilon.$$

□

Proof of Lemma 4.1. The proof of Lemma 4.1 is similar to the proof of Lemma 4.2. Firstly, since $u(x, y, \beta)$ is uniformly continuous on $[0, 1] \times [0, 1]$, given $\epsilon > 0$, there exists $m_1 > 0$ such that

$$\sup_{\substack{|s-t| < \frac{1}{m_1} \\ s, t, y \in [0, 1]}} |u(s, y, \beta) - u(t, y, \beta)| < \frac{\epsilon}{6}.$$

Hence, for any $|s - t| < \frac{1}{m_1}$ with $s, t \in [0, 1]$, we have

$$\begin{aligned} & \left| \int_{y_1}^{y_2} u(s, y, \beta) dy - \int_{y_1}^{y_2} u(t, y, \beta) dy \right| \\ & \leq \int_{y_1}^{y_2} |u(s, y, \beta) - u(t, y, \beta)| dy \\ & \leq \int_0^1 |u(s, y, \beta) - u(t, y, \beta)| dy \\ & < \frac{\epsilon}{6}. \end{aligned} \tag{63}$$

Then, choose an $m > 2m_1$ such that

$$\left| e^{\frac{\beta}{m}} - 1 \right| < \frac{\epsilon}{12}, \quad \left| e^{-\frac{\beta}{m}} - 1 \right| < \frac{\epsilon}{12}.$$

Next, since $\lim_{n \rightarrow \infty} q_n^n = e^{-\beta}$, there exists $N_1 > 0$ such that for any $n > N_1$,

$$\left| e^{\frac{\beta}{m}} - q_n^{-\frac{n}{m}} \right| < \frac{\epsilon}{12}, \quad \left| e^{-\frac{\beta}{m}} - q_n^{\frac{n}{m}} \right| < \frac{\epsilon}{12}.$$

By triangle inequality, for any $n > N_1$,

$$\max \left(\left| 1 - q_n^{\frac{n}{m}} \right|, \left| 1 - q_n^{-\frac{n}{m}} \right| \right) < \frac{\epsilon}{6}.$$

Next, define the $m + 1$ sequences $\{a_n^{(k)}\}$, $0 \leq k \leq m$, as in (57). By (59) and Lemma 5.7, for any $n > \max(m, N_1)$ and any $i \in [n]$ with corresponding k defined in (58), we have

$$\left| \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(i)}{n})) - \mu_{n,q_n}(\mathbb{1}_A(\frac{\pi(a_n^{(k)})}{n})) \right| \tag{64}$$

$$\begin{aligned} &\leq \max \left(\left| 1 - q_n^{\frac{n}{m}} \right|, \left| 1 - q_n^{-\frac{n}{m}} \right| \right) \\ &< \frac{\epsilon}{6}. \end{aligned}$$

Secondly, by the definition of $a_n^{(k)}$ in (57), it is easily seen that

$$\frac{kn}{m} \leq a_n^{(k)} \leq \frac{kn}{m} + 1.$$

Thus, for any $n > m$ and any $i \in [n]$ with corresponding k defined in (58), we have

$$\begin{aligned} &\frac{kn}{m} \leq a_n^{(k)} \leq i < a_n^{(k+1)} \leq \frac{(k+1)n}{m} + 1 \\ \Rightarrow &\frac{k}{m} \leq \frac{i}{n} \leq \frac{k+1}{m} + \frac{1}{n} \\ \Rightarrow &\left| \frac{i}{n} - \frac{k}{m} \right| \leq \frac{1}{m} + \frac{1}{n} < \frac{2}{m} < \frac{1}{m_1}. \end{aligned}$$

Hence, by (63), for any $n > m$ and any $i \in [n]$ with corresponding k defined in (58), we have

$$\left| \int_{y_1}^{y_2} u\left(\frac{i}{n}, y, \beta\right) dy - \int_{y_1}^{y_2} u\left(\frac{k}{m}, y, \beta\right) dy \right| < \frac{\epsilon}{6}. \quad (65)$$

Thirdly, since $\lim_{n \rightarrow \infty} \frac{a_n^{(k)}}{n} = \frac{k}{m}$ for any $0 \leq k \leq m$, by Theorem 1.2, there exists $N_1 > 0$ such that, for any $n > N_1$ and any $0 \leq k \leq m$,

$$\left| \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right) \right) - \int_{y_1}^{y_2} u\left(\frac{k}{m}, y, \beta\right) dy \right| < \frac{\epsilon}{3}. \quad (66)$$

Therefore, for any $n > \max(m, N_1, N_2)$ and any $i \in [n]$ with corresponding k defined in (58), we have

$$\begin{aligned} &\left| \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) - \int_{y_1}^{y_2} u\left(\frac{i}{n}, y, \beta\right) dy \right| \\ &\leq \left| \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(i)}{n} \right) \right) - \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right) \right) \right| \\ &\quad + \left| \mu_{n, q_n} \left(\mathbb{1}_A \left(\frac{\pi(a_n^{(k)})}{n} \right) \right) - \int_{y_1}^{y_2} u\left(\frac{k}{m}, y, \beta\right) dy \right| \\ &\quad + \left| \int_{y_1}^{y_2} u\left(\frac{k}{m}, y, \beta\right) dy - \int_{y_1}^{y_2} u\left(\frac{i}{n}, y, \beta\right) dy \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} < \epsilon. \end{aligned}$$

The last inequality follows from (64), (65) and (66). \square

6 Discussion and open questions

One question which arises in the proof of Theorem 1.3 is that does Theorem 1.3 also holds in more general settings. Specifically, we are given two sequences of probability measures $\{\mathbb{P}_n^{(1)}\}_{n=1}^\infty, \{\mathbb{P}_n^{(2)}\}_{n=1}^\infty$ such that, for $j = 1, 2$, $\mathbb{P}_n^{(j)}$ is a probability measure on S_n . Also let $\rho_1(x, y)$ and $\rho_2(x, y)$ be two density functions on $[0, 1] \times [0, 1]$ such that, for any $\epsilon > 0$ and any $j = 1, 2$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n^{(j)} \left(\left| \frac{1}{n} \sum_{i=1}^n f \left(\frac{i}{n}, \frac{\pi(i)}{n} \right) - \int_0^1 \int_0^1 f(x, y) \rho_j(x, y) dx dy \right| > \epsilon \right) = 0,$$

for any continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$. Let \mathbb{P}_n denote the probability measure on $S_n \times S_n$ such that $\mathbb{P}_n((\pi, \tau)) = \mathbb{P}_n^{(1)}(\pi) \cdot \mathbb{P}_n^{(2)}(\tau)$, i.e. \mathbb{P}_n is the product measure of $\mathbb{P}_n^{(1)}$ and $\mathbb{P}_n^{(2)}$. Then, does the following holds? For any continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left(\left| \frac{1}{n} \sum_{i=1}^n f \left(\frac{i}{n}, \frac{\tau \circ \pi(i)}{n} \right) - \int_0^1 \int_0^1 f(x, y) \rho(x, y) dx dy \right| > \epsilon \right) = 0,$$

where $\rho(x, y) := \int_0^1 \rho_1(x, t) \cdot \rho_2(t, y) dt$.

The proofs of Lemma 4.1 and Lemma 4.2, which are the main steps in showing Theorem 1.3, rely on special properties of the Mallows measure. Specifically, Theorem 1.2 and Proposition 5.2 play the key roles in our proofs of Lemma 4.1 and Lemma 4.2.

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